COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Balanced Use of Resources in Computations

Master's Thesis

2013

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FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Balanced Use of Resources in Computations

Master's Thesis

Study Programme: Informatics
Branch of Study: 9.2.1. Informatics
Department: Department of Computer Science
Supervisor: prof. RNDr. Branislav Rovan, PhD.

Bratislava, 2013

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ZADANIE ZÁVEREČNEJ PRÁCE

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Študijný program: informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor: 9.2.1. informatika
Typ záverečnej práce: diplomová
Jazyk záverečnej práce: anglický
Sekundárny jazyk: slovenský

Názov: Rovnomerné využívanie prostriedkov vo výpočtoch

Cieľ:
- Zovšeobecniť definície konečných automatov s rovnomerným využívaním prostriedkov aj na niektoré ďalšie často používané deterministické výpočtové modely.
- Zjednotiť teórie stavovo a prechodovo vyvážených deterministických konečných automatov.
- Započať štúdium vyváženosti pre nejaký výpočtový model akceptujúci neregulárne jazyky (napr. pre deterministické jednopočítadlové automaty).

Vedúci: prof. RNDr. Branislav Rovan, PhD.
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Dátum zadania: 03.11.2011

Dátum schválenia: 03.11.2011 prof. RNDr. Branislav Rovan, PhD.
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Field of Study:	9.2.1. Computer Science, Informatics
Type of Thesis:	Diploma Thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Balanced Use of Resources in Computations

Aim:
- To generalize the definitions of finite automata with balanced use of resources to some other frequently used deterministic computation models.
- To unify the theories of state-equiloaded and transitionally equiloaded deterministic finite automata.
- To initiate a study of equiloadedness for some computation model accepting nonregular sets (e.g., for deterministic one-counter automata).

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Student

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I hereby declare that I wrote this thesis by myself, only with the help of the referenced literature, under the careful supervision of my thesis advisor.

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Acknowledgement

I would like to express my immense gratitude to my thesis advisor, Professor Branislav Rovan, for introducing me to a fruitful area of research and kindly guiding me on the winding routes of writing this thesis. I deeply acknowledge his willingness to help always when needed, and our numerous conversations that offered me more insight into the nature of computer science than any regular class could.

Abstrakt

Autor:	Peter Kostolányi
Názov práce:	Rovnomerné využívanie prostriedkov vo výpočtoch (Balanced Use of Resources in Computations)
Univerzita:	Univerzita Komenského v Bratislave
Fakulta:	Fakulta matematiky, fyziky a informatiky
Katedra:	Katedra informatiky
Vedúci práce:	prof. RNDr. Branislav Rovan, PhD.
Rozsah:	94 strán

Práca sa zaoberá rovnomerným využívaním prostriedkov v deterministických sekvenčných výpočtoch. Predložených je niekoľko definícií automatov s rovnomerným využívaním prostriedkov (vyvážených automatov). Tieto definície sú formulované pre abstraktné deterministické automaty, t.j. pre abstrakciu automatov definovanú pre tento účel. Vyváženosť je potom predmetom štúdia pre tri špeciálne prípady abstraktných deterministických automatov: pre deterministické konečné automaty, deterministické konečné automaty s prechodmi na prázdne slovo a deterministické jednopočítadlové automaty. V práci je dokázaných viacero charakterizácií tried vyvážených automatov. Práca sa tiež zaoberá triedami jazykov akceptovaných vyváženými automatmi.

KĽÚČOVÉ SLOVÁ: rovnomerné využívanie prostriedkov, vyváženosť, vyvážený automat, abstraktný deterministický automat.

Abstract

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Number of Pages:	94

Balanced use of resources in deterministic sequential computations is studied in this thesis. Several definitions of automata with balanced use of resources (equiloaded automata) are presented. These definitions are presented for abstract deterministic automata, i.e., an abstraction of automata devised for this purpose. Equiloadedness is then studied for three particular cases of abstract deterministic automata: deterministic finite automata, deterministic finite automata with ε-transitions, and deterministic one-counter automata. Several characterizations of families of equiloaded automata are proved. The families of languages accepted by equiloaded automata are also considered in the thesis.

KEYWORDS: balanced use of resources, equiloadedness, equiloaded automaton, abstract deterministic automaton.

Contents

Introduction

In this thesis, we shall study the concept of balanced use of resources in deterministic sequential computations. The problem of balancing the use of computational resources arises in several practical applications. However, up to now similar problems have received enough theoretical attention only in a parallel setting – the problem of balancing the load of computational tasks among several processors has been studied extensively. On the contrary, we are interested in sequential computations in this thesis, and our research may be viewed as a theoretical attempt to study computations with balanced use of multiple parts of a single processor.

Our approach to this problem is to study deterministic sequential models of computation and to come up with several possible definitions of automata with balanced use of resources. The resources considered are states and transitions. We shall call such automata *equiloaded*. We shall be interested in the properties of several families of equiloaded automata, and in the properties of families of languages accepted by these automata (*equiloaded languages*).

In this thesis, we continue the research initiated in several previous works. In [27] and [28], the balanced use of states in deterministic finite automata has been studied. In [25], we have studied the balanced use of transitions in deterministic finite automata.

The definitions of equiloadedness studied in this thesis are based on the definitions used in these previous works, however are significantly generalized. This generalization is twofold: first, the definitions presented in this thesis do not apply to deterministic finite automata only, but are defined for an abstract model of computation (*abstract deterministic automata*) that allows us to define equiloadedness for a variety of computational models at once. The second generalization has to do with sets of computations that we are concerned with while studying equiloadedness.

The main goals of this thesis therefore can be summarized as follows. First of all, we aim to present several sensible definitions of equiloadedness, suitable for diverse types of deterministic automata. Second, some of the definitions of state-equiloaded deterministic finite automata, used in [27] and [28], slightly differ from the definitions of transition-equiloaded deterministic finite automata, used in [25]. Thus, one of the aims of this thesis is to unify both theories by showing equivalence of certain definitions and by studying some aspects of equiloaded deterministic finite automata that have not been studied yet. Finally, since deterministic finite automata are the only model of computation, for which equiloadedness has been studied so far, we extend our study also to some other models of computation. In this thesis, we shall concentrate on equiloaded deterministic finite automata with ε-transitions (we shall observe that the possibility of ε-transitions adds some computational power to equiloaded deterministic finite automata) and initiate the study of equiloaded deterministic one-counter automata.

The structure of the thesis is as follows:

- *In Chapter 1*, after stating some preliminary definitions, we shall define an abstract model of computation inspired by abstract families of automata of S. Ginsburg [14] – an *abstract deterministic automaton*. We shall observe that some widely studied models of computation, e.g., deterministic finite automata, deterministic one-counter automata, or deterministic pushdown automata, are special cases of abstract deterministic automata.

  Further in this chapter, we shall present definitions of equiloadedness used in this thesis. These definitions will be stated for abstract deterministic automata, and thus will apply

to a large variety of deterministic models of computation. Finally, we shall examine some properties of equiloadedness that hold for abstract deterministic automata in general.

- *In Chapter 2*, we shall focus on equiloaded deterministic finite automata. We shall unify the theories from [27], [28], and [25] and prove some new results. Moreover, we shall extend the theory to deterministic finite automata with ε-transitions.
- *In Chapter 3*, we shall initiate the study of equiloaded deterministic one-counter automata.

In order to preserve a reasonable length of this thesis, we have been forced to omit some of the material. Some proofs have been omitted or partially omitted in this thesis, and some explanatory examples have been excluded. Moreover, some of the more advanced mathematics (e.g., the Perron-Frobenius theory) is used without providing an introduction to the topic. However, the reader may find the omitted material in the report [26] that is in fact an extended version of this thesis.

Chapter 1

Definitions and Basic Abstract Results

The main aim of this chapter is to present definitions of equiloadedness that we shall use in this thesis and to examine some of their basic properties that hold independently of a particular model of computation. We shall follow two conceptually different approaches to the definition of balacned use of resources. First, we shall define the concept of *strict $\mathcal{S}$-equiloadedness*, where $\mathcal{S}$ specifies the set of computation paths considered. Next, we shall define a slightly more involved concept of $\mathcal{S}$-equiloadedness and a related concept of weak $\mathcal{S}$-equiloadedness.

The chapter is structured as follows. In Section 1.1, we shall present some preliminary definitions that we shall use in this thesis. Most importantly, we shall present definitions of computational models that we shall study in this thesis: deterministic finite automata and deterministic one-counter automata. There is a need for including these definitions, since there are different variants of these models commonly used in literature. However, despite these differences usually do not matter, we shall observe that in the case of equiloaded automata, minor details in definitions of computational models may have significant consequences for their computational power.

In Section 1.2, we shall define an abstraction of deterministic automata with a one-way input tape (inspired by the concept of abstract families of automata [14]), the *abstract deterministic automata* (ADA). We shall observe that both deterministic finite automata and deterministic one-counter automata are special cases of ADA. The main reason for introducing this abstraction is that it enables us to define equiloadedness for a variety of computational models at once. That is, we shall present only one definition of each kind of equiloadedness that will apply to all models of computation considered in this thesis.

In Section 1.3, we shall briefly introduce some basic quantities that will serve as a cornerstone of our definitions of equiloadedness.

In Section 1.4, we shall define strictly $\mathcal{S}$-equiloaded automata. We shall state the definition independently of a particular computational model, i.e., for abstract deterministic automata. Here, $\mathcal{S}$ is a parameter specifying the set of computation paths considered.

In Section 1.5, we shall define $\mathcal{S}$-equiloaded automata. By slightly relaxing the conditions imposed in this definition, we shall obtain the related definition of weakly $\mathcal{S}$-equiloaded automata. Similarly as in the case of strict $\mathcal{S}$-equiloadedness, the definition is stated independently of a particular model of computation, and $\mathcal{S}$ specifies the sets of computation paths considered.

In Section 1.6, we shall examine some relations between the families of equiloaded languages that hold in general for every model of computation that is a special case of ADA.

Finally, in Section 1.7, we shall introduce a concept of prefix-dense languages. This concept will serve as a useful tool for proving that a given language is not strictly $\mathcal{S}$-equiloaded, for any computational model that is a special case of ADA.

1.1 Preliminaries

In this section, we shall briefly present some basic definitions used in this thesis. First, let us present our definition of deterministic finite automata. We shall be concerned with two different variants of deterministic finite automata: deterministic finite automata without ε-transitions (DFA) and deterministic finite automata with the possibility of deterministic ε-transitions (DFAε).

Definition 1.1.1 A *deterministic finite automaton with ε-transitions* (DFAε) A is a five-tuple $A = (K, \Sigma, \delta, q_0, F)$, where K is a nonempty finite set of states, Σ is an alphabet, $\delta : K \times (\Sigma \cup \{\varepsilon\}) \to K$ is a *partial* transition function that can be defined for (q, ε), q in K, only if $\delta(q, c)$ is not defined for any c in Σ, q_0 in K is the initial state, and $F \subseteq K$ is the set of accepting states. A *deterministic finite automaton without ε-transitions* (DFA) is a DFAε with a transition function not defined on ε.

Thus, there are two important facts that are necessary to keep in mind about our definition of deterministic finite automata. First, we are concerned with deterministic finite automata with a partial transition function, i.e., the transition function need not be defined for all possible inputs. Second, we are concerned with two different variants of deterministic finite automata, depending on if deterministic ε-transitions are allowed or not. Further, let us note that every DFA is at the same time a DFAε. Thus, if a theorem is stated for all DFAε, it holds also for all DFA. However, this is not true when talking about, e.g., the corresponding families of equiloaded languages.

The definitions of a configuration, a computation step and of the accepted language are standard. Moreover, we shall define a *transition* of a DFAε $A = (K, \Sigma, \delta, q_0, F)$ to be an arbitrary triple (p, c, q) in $K \times (\Sigma \cup \{\varepsilon\}) \times K$, such that $\delta(p, c) = q$. We shall denote the set of all transitions of the automaton A by D_A, or D.

Further, we shall define a *computation path* of the automaton A to be an arbitrary sequence of transitions corresponding to some computation of the automaton A. The formal definition is as follows.

Definition 1.1.2 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε. A *computation path* of the automaton A is a *finite* sequence $\gamma = \{(q_1, c_1, q_1'), \ldots, (q_n, c_n, q_n')\}$ of transitions in D, such that $q_1 = q_0$, and the property $q_{k+1} = q_k'$ holds for $k = 1, \ldots, n-1$. The number n is referred to as the *length* $|\gamma|$ *of the computation path* γ. A computation path γ is said to be *accepting*, if the state q_n' is accepting, i.e., if q_n' is in F.

We define the *graphical representation* of the automaton A to be a Σ-weighted digraph (with the possibility of multiple edges and loops) with the set of vertices corresponding to K, and with a c-weighted arc from a vertex p to a vertex q, if and only if (p, c, q) is in D. We shall define the *transition matrix* of the automaton A to be the adjacency matrix of its graphical representation. More formally, we shall define the transition matrix as follows.

Definition 1.1.3 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε. Let $K = \{q_0, \ldots, q_{m-1}\}$. The *transition matrix* of the automaton A is an $m \times m$ matrix

$$\Delta_A := \begin{pmatrix} d_{0,0} & d_{0,1} & \ldots & d_{0,m-1} \\ d_{1,0} & d_{1,1} & \ldots & d_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m-1,0} & d_{m-1,1} & \ldots & d_{m-1,m-1} \end{pmatrix},$$

where $d_{i,j}$ is defined by

$$d_{i,j} = \left|\{c \in \Sigma \cup \{\varepsilon\} \mid \delta(q_i, c) = q_j\}\right|,$$

for all i, j in $\{0, \ldots, m-1\}$. We shall omit the subscript A when A is clear from the context.

Now, let us define deterministic one-counter automata. We shall use a definition, in which a counter is represented by a single nonnegative integer (alternatively, a counter may be represented by a pushdown store over an unary alphabet).

Definition 1.1.4 A *deterministic one-counter automaton* (DOCA) A is a five-tuple $A = (K, \Sigma, \delta, q_0, F)$, where K is a nonempty finite set of states, Σ is an alphabet, $\delta : K \times (\Sigma \cup \{\varepsilon\}) \times \{0,1\} \to K \times \{-1, 0, 1\}$ is a deterministic (ε-transition may be defined only if there is not any other transition defined for given p in K and t in $\{0,1\}$) *partial* transition function, such that if $\delta(p, c, 0) = (q, r)$ for some p, q in K and c in $\Sigma \cup \{\varepsilon\}$, then r is in $\{0,1\}$, q_0 in K is the initial state, and $F \subseteq K$ is the set of accepting states.

We assume that the reader is familiar with the idea behind one-counter automata. Thus, we shall omit the standard definitions of a confinguration, a computation step, an accepted language, and of a language accepted by empty memory, as well as definitions of a transition and a computation path that are analogous to the definitions for DFA and DFAε. The reader may find these definitions in our report [26].

1.2 Abstract Deterministic Automata

In this section, we shall define a new abstract model of computation that is meant to serve, for the purposes of this thesis, as a simple generalization of several well-known and extensively studied deterministic computation models. We shall call these abstract automata *Abstract Deterministic Automata* (ADA).

Before we present a formal definition of abstract deterministic automata, let us briefly point out the importance of such a construction. In the next sections of this chapter, we shall define several types of equiloaded automata. For such definitions, some degree of independence from the computation model is a highly desired property. We would prefer to avoid separate definitions for each model of computation, and to devise, for each type of equiloadedness, one general definition applicable to a variety of interesting models of computation (e.g., deterministic finite automata, deterministic one-counter automata, deterministic pushdown automata, some variants of deterministic Turing machines, etc.). Since we shall be only concerned with models of computation that are special cases of ADA in this thesis, we shall consider definitions stated for ADA to be independent from a particular computational model.

Several other abstractions of automata are known up to date, for instance *Abstract Families of Automata* (AFA) [14], or *Balloon Automata* [20] [21]. However, both of these constructions are devised in order to guarantee certain properties of families of languages accepted by such automata and hence are, for the purposes of this thesis, unnecessarily complicated. Our definition of abstract deterministic automata does not guarantee these properties of accepted families of languages, but on the other hand, the definition is considerably simpler.

We shall define abstract deterministic automata as an abstraction of *deterministic one-way* automata. Therefore the requirements imposed on abstract deterministic automata are as follows:

- An ADA has a one-way input tape with ε-transitions allowed (that is, the input tape is read by the automaton exactly as in the case of, e.g., deterministic pushdown automata), and some kind of auxiliary memory storage.

- Although the auxiliary memory storage can acquire a possibly infinite number of memory contents, the transition function of the ADA distinguishes only a finite number of outputs of reading the auxiliary memory. The transitions of the ADA are executed based only on the information obtained by this output. For instance, in deterministic pushdown automata, the number of possible words on the pushdown store is infinite. However, the transition function distinguishes only between characters on the top of the pushdown store, i.e., members of a finite alphabet.

- An ADA is always deterministic – that is, for each state, input symbol, and output of reading the auxiliary memory, at most one transition can be executed.

This leads us to the formal definition that is as follows.

Definition 1.2.1 An *abstract deterministic automaton* (ADA) is a nine-tuple $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$, where K is a nonempty finite set of states, Σ is an input alphabet, G is a (finite or infinite) set of possible auxiliary memory contents, H is a finite set of outputs of reading the auxiliary memory, $\zeta : G \to H$ is a read function, Z in G is an initial content of the auxiliary memory, $\delta : K \times (\Sigma \cup \{\varepsilon\}) \times H \to K \times G^G$ is a *partial* transition function, q_0 in K is the initial state, and $F \subseteq K$ is the set of accepting states. Moreover, the transition function δ must be deterministic, i.e., the property

$$\forall p \in K \forall h \in H : (\exists (q, \mu) \in K \times G^G : \delta(p, \varepsilon, h) = (q, \mu)) \Rightarrow$$
$$\neg(\exists a \in \Sigma \exists (q', \mu') \in K \times G^G : \delta(p, a, h) = (q', \mu')).$$

is required to hold.

Let us explain the way the transition function δ is defined in the above definition. The transition function of an abstract deterministic automaton takes three arguments: a state, an input symbol (or ε, i.e., ε-transitions are allowed), and an output of reading the auxiliary memory. The transition function outputs a new state, and some transformation of the auxiliary memory μ. Every such transformation is a function from G to G, i.e., a member of the set G^G. We do not impose any restriction on these transformations, however we shall be interested only in cases, where these transformations are reasonable. The implication that is required to hold for the transition function δ ensures that only deterministic ε-transitions are allowed. Let us proceed by the definition of a configuration.

Definition 1.2.2 Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be an abstract deterministic automaton. A *configuration* of the automaton A is a triple (q, w, g) in $K \times \Sigma^* \times G$, where q is a state, w is an unread part of the input word, and g is a content of the auxiliary memory.

Now, let us define a computation step. This definition is more-or-less standard, given an intuitive idea of ADA presented above. An automaton is supposed to make a computation step from a given configuration (p, cw, g), where p is a state, c is a symbol (or the empty word ε) to be read from the input, and g is a content of the auxiliary memory. To make a computation step to a second configuration (q, w, g'), a transition function δ, given the state p, the symbol c, and the output $\zeta(g)$ after reading the content of the auxiliary memory g, has to return the state q and an auxiliary memory transformation μ, such that it transforms g into g', i.e., $\mu(g) = g'$. The definition of the computation step is thus as follows.

Definition 1.2.3 Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be an abstract deterministic automaton. A *computation step* of the automaton A is a binary relation $\vdash_A$ on configurations of the automaton A defined as follows:

$$(p, cw, g) \vdash_A (q, w, g') \iff \delta(p, c, \zeta(g)) = (q, \mu),$$

where p, q are in K, c is in $\Sigma \cup \{\varepsilon\}$, w is in Σ^*, g, g' are in G, and $\mu : G \to G$ is a mapping, such that $\mu(g) = g'$. If A is clear from the context, we shall write $\vdash$ instead of $\vdash_A$. By $\vdash_A^*$, we shall denote the reflexive and transitive closure of the relation $\vdash_A$.

Now, we may define the language accepted by a given abstract deterministic automaton. For several widely studied models of computation that are special cases of ADA, as for instance for deterministic one-counter automata or deterministic pushdown automata,[1] there is more then one mode of acceptation extensively studied: for instance, we may consider the language accepted by the accepting state, the language accepted by empty memory, etc. For ADA, we shall define two modes of acceptation: acceptation by the accepting state and acceptation by memory,

[1] We consider the observation that these models of computation are indeed special cases of ADA, to be obvious. However, in the end of this section, we shall present some examples, in which we shall formally model some of the widely known models of computation in terms of ADA.

with a possibility to model the acceptation by empty memory by the latter (at least for some special cases of ADA – for DFA for instance, acceptation by empty memory does not make much sense).

Definition 1.2.4 Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be an abstract deterministic automaton. The *language* accepted by the automaton A is the set

$$L(A) = \{w \in \Sigma^* \mid \exists (q, g) \in F \times G : (q_0, w, Z) \vdash_A^* (q, \varepsilon, g)\}.$$

Definition 1.2.5 Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be an abstract deterministic automaton, let $M \subseteq G$ be a set of memory contents. The *language* accepted by the automaton A *by memory* in M is a set

$$N_M(A) = \{w \in \Sigma^* \mid \exists (q, g) \in K \times M : (q_0, w, Z) \vdash_A^* (q, \varepsilon, g)\}.$$

For some special cases of ADA, as for instance deterministic one-counter automata or deterministic pushdown automata, we define a special memory content 0_A, representing the empty memory. For such models of computation, we define the *language accepted by empty memory*, $N(A)$, to be the language accepted by memory in $\{0_A\}$. We shall call the special cases of ADA, for which we define the content 0_A, the models of computation *with the ability to accept by empty memory*. The definition of a transition is similar as, e.g., in the case of DFA.

Definition 1.2.6 Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be an abstract deterministic automaton. A *transition* of the automaton A is a five-tuple

$$(q, c, h, q', \mu) \in K \times (\Sigma \cup \{\varepsilon\}) \times H \times K \times G^G,$$

such that $\delta(q, c, h) = (q', \mu)$. We shall denote the set of all transitions of the automaton A by D_A (or by D, if A is clear from the context).

Finally, let us present the last definition related to abstract deterministic automata: the definition of a computation path. Similarly as in the case of DFA, a computation path is an arbitrary finite sequence of transitions corresponding to some computation of the automaton. This intuitive idea is formally expressed as follows.

Definition 1.2.7 Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be an abstract deterministic automaton. A *computation path* of the automaton A is a *finite* sequence

$$\gamma = \{(q_1, c_1, h_1, q_1', \mu_1), \ldots, (q_n, c_n, h_n, q_n', \mu_n)\} \in D_A^n$$

of n transitions of the automaton A, such that $q_1 = q_0$, and the following two properties hold:

(i) For $k = 1, \ldots, n-1$, the property $q_{k+1} = q_k'$ holds.

(ii) There is a sequence of auxiliary memory contents,

$$\mathbf{g} = \{g_1, \ldots, g_{n+1}\} \in G^{n+1},$$

such that:

1. $g_1 = Z$.
2. For $k = 1, \ldots, n$, the property $h_k = \zeta(g_k)$ holds.
3. For $k = 1, \ldots, n$, the property $\mu_k(g_k) = g_{k+1}$ holds.

The *length* $|\gamma|$ *of the computation path* γ is the number $|\gamma| = n$. A computation path γ is said to be *accepting*, if q_n' is in F. Moreover, if A has an ability to accept by empty memory, a computation path γ is said to be *accepting by empty memory*, if $g_{n+1} = 0_A$.

From the presented definitions, it should be clear that several extensively studied models of computation, including DFA, DFAε and DOCA, can be viewed as a special case of abstract deterministic automata. In what follows, we shall present formal constructions establishing this fact. For the sake of clarity of exposition, we shall avoid formal definitions of families of automata, and related concepts. Moreover, we shall use an intuitive notion of *isomorphism of automata*. We shall say that two automata are isomorphic, if all of their computations are isomorphic. However, we shall not formally define this notion of isomorphism, and shall rely on intuition. The notion of automata isomorphism is important, since automata defined in Section 1.1 are formally not the same tuples as abstract deterministic automata. However, we can clearly see if an abstract deterministic automaton behaves in its essence as one of the models of computation defined in Section 1.1.

Example 1.2.8 First, we shall show that deterministic finite automata and deterministic finite automata with ε-transitions are the special cases of abstract deterministic automata.

Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton with or without ε-transitions. Let us define the abstract deterministic automaton $A' = (K', \Sigma', G', H', \zeta', Z', \delta', q_0', F')$ isomorphic to A as follows: let $K' = K$, $\Sigma' = \Sigma$, $G' = \{\circ\}$, $H' = \{\circ\}$. Next, the read function $\zeta' : G' \to H'$ is defined by $\zeta'(\circ) = \circ$, and $Z' = \circ$. The transition function $\delta' : K' \times (\Sigma' \cup \{\varepsilon\}) \times H' \to K' \times G'^{G'}$ is defined by

$$\delta'(p, c, \circ) = (q, Id) \iff \delta(p, c) = q,$$

where p, q are in $K = K'$, c is in $\Sigma \cup \{\varepsilon\} = \Sigma' \cup \{\varepsilon\}$, and $Id : G' \to G'$ is an identical mapping on G' (i.e., $Id(\circ) = \circ$). Finally, let us define $q_0' = q_0$, and $F' = F$.

Automata A and A' are clearly isomorphic. Moreover, A' has ε-transitions iff A has ε-transitions. That is, deterministic finite automata and deterministic finite automata with ε-transitions are both special cases of abstract deterministic automata.

Example 1.2.9 Now, we shall show that deterministic one-counter automata can be viewed as a special case of abstract deterministic automata. Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic one-counter automaton. We shall construct the isomorphic abstract deterministic automaton $A' = (K', \Sigma', G', H', \zeta', Z', \delta', q_0', F')$ as follows: let us define $K' = K$, $\Sigma' = \Sigma$, $G' = \mathbb{N}$, $H' = \{0, 1\}$. The read function $\zeta' : G' \to H'$ will be defined by

$$\forall n \in \mathbb{N} : \zeta'(n) = \mathrm{sgn}(n).$$

Next, $Z' = 0$, and the transition function $\delta' : K' \times (\Sigma' \cup \{\varepsilon\}) \times H' \to K \times G'^{G'}$ will be defined by

$$\delta'(p, c, t) = (q, \mu_r) \iff \delta(p, c, t) = (q, r),$$

where p, q are in $K = K'$, c is in $\Sigma \cup \{\varepsilon\} = \Sigma' \cup \{\varepsilon\}$, t is in $\{0, 1\}$, r is in $\{-1, 0, 1\}$, and $\mu_r : G' \to G'$ is defined by

$$\forall n \in \mathbb{N} : \mu_r(n) = \max\{0, n + r\}.$$

Since the definition of deterministic one-counter automata ensures that if $t = 0$ then $r \geq 0$, the function μ_r always adds r to the counter (auxiliary memory). Finally, let us define $q_0' = q_0$, and $F' = F$.

Moreover, we shall define $0_{A'} = 0$. Thus, deterministic one-counter automata are able to accept by empty memory.

Thus, if we define concepts or state theorems for abstract deterministic automata in the rest of this thesis, we shall always keep in mind that the same concepts or theorems apply to all types of automata that we are concerned with in this thesis.

Finally in this section, let us introduce a notation for some special sets of computation paths that we shall use in this thesis.

Notation 1.2.10 Let A be an abstract deterministic automaton.

a) $\mathrm{Comp}(A)$ denotes the (possibly infinite) set of all computation paths of A.

b) $\mathrm{Comp}(A, n)$ denotes the (always finite) set of all computation paths of A of length n in $\mathbb{N}$.

c) $\mathrm{Comp}(A, \leq n)$ denotes the (always finite) set of all computation paths of A of length less than or equal to n in $\mathbb{N}$.

d) $\mathrm{Acc}(A)$ denotes the (possibly infinite) set of all accepting computation paths of A.

e) $\mathrm{Acc}(A, n)$ denotes the (always finite) set of all accepting computation paths of A of length n in $\mathbb{N}$.

f) $\mathrm{Acc}(A, \leq n)$ denotes the (always finite) set of all accepting computation paths of A of length less than or equal to n in $\mathbb{N}$.

Moreover, if A is an abstract deterministic automaton with an ability to accept by empty memory, we shall use the following notation:

g) $\mathrm{eAcc}(A)$ denotes the (possibly infinite) set of all computation paths of A accepting by empty memory.

h) $\mathrm{eAcc}(A, n)$ denotes the (always finite) set of all computation paths of A of length n in $\mathbb{N}$ accepting by empty memory.

i) $\mathrm{eAcc}(A, \leq n)$ denotes the (always finite) set of all computation paths of A of length less than or equal to n in $\mathbb{N}$ accepting by empty memory.

1.3 Basic Quantities

In this section, we shall briefly introduce some notation that will serve as a cornerstone of our definitions of equiloadedness, presented in the later sections of this chapter.

Notation 1.3.1 Let A be an ADA with the set of states K, and the set of transitions D. Let γ be a computation path of the automaton A, q in K be a state, and e in D be a transition. By the symbol $\#_A[q, \gamma]$, we shall denote the number of uses of the state q in the computation path γ. Similarly, by the symbol $\#_A[e, \gamma]$, we shall denote the number of uses of the transition e in the computation path γ.

Further, let Cmp be a *finite* set of computation paths of the automaton A. We shall use the notation

$$\#_A[q, Cmp] = \sum_{\gamma \in Cmp} \#_A[q, \gamma], \qquad \#_A[e, Cmp] = \sum_{\gamma \in Cmp} \#_A[e, \gamma].$$

If A is clear from the context, we shall omit the subscript A from the notation.

1.4 Strict $\mathcal{S}$-Equiloadedness

In this section, we shall present a definition of *strict $\mathcal{S}$-equiloadedness*, representing a first possible viewpoint on the nature of balanced use of resources. Informally, an automaton is strictly state-$\mathcal{S}$-equiloaded (strictly transition-$\mathcal{S}$-equiloaded), if its states (transitions) are used approximately the same number of times in every computation path from some set specified by $\mathcal{S}$. Thus, $\mathcal{S}$ is a parameter that specifies the set of computation paths considered.

Similar definitions have been used in [27], [28], and [25], in order to study balanced use of resources in DFA. However, the definition used in this thesis is more general. First, it is stated for ADA instead of DFA (this is also a reason for certain formal differences between the present definition and the definitions from the previous works). Second, the possibility to choose the

parameter $\mathcal{S}$ is new – the definitions from the previous works have been concerned solely with the fixed set of all accepting computation paths.

The set $\mathcal{S}(A)$ of computation paths considered for a given automaton A is required to be specified for all ADA at once. Thus, $\mathcal{S}$ is a function defined on the family of all abstract deterministic automata that returns some set of computation paths of its input.

The formal definition is as follows.

Definition 1.4.1 Let $\mathcal{S}$ be a function that for each ADA A returns some set $\mathcal{S}(A)$ of computation paths of the automaton A. Let A be an ADA with the set of states K and the set of transitions D.

a) A is said to be *strictly state-$\mathcal{S}$-equiloaded*, if a real constant η in $\mathbb{R}$ exists, such that for all computation paths γ in $\mathcal{S}(A)$ and for all pairs of states p, q in K, the property

$$|\#[p, \gamma] - \#[q, \gamma]| \leq \eta$$

holds.

b) A is said to be *strictly transition-$\mathcal{S}$-equiloaded*, if a real constant η in $\mathbb{R}$ exists, such that for all computation paths γ in $\mathcal{S}(A)$ and for all pairs of transitions e, f in D, the property

$$|\#[e, \gamma] - \#[f, \gamma]| \leq \eta$$

holds.

Further, let x in $\{\mathrm{DFA}, \mathrm{DFA}\varepsilon, \mathrm{DOCA}, \ldots\}$ be an abbreviation of some model of computation that is a special case of ADA. A language L is said to be a *strictly state-$\mathcal{S}$-equiloaded x-language*, if a strictly state-$\mathcal{S}$-equiloaded ADA A of type x exists, such that $L(A) = L$. Similarly, a language L is said to be a *strictly transition-$\mathcal{S}$-equiloaded x-language*, if a strictly transition-$\mathcal{S}$-equiloaded ADA A of type x exists, such that $L(A) = L$.

We shall denote the family of strictly state-$\mathcal{S}$-equiloaded x-languages by $\mathscr{L}_{K-SEQ-x}(\mathcal{S})$. The family of all strictly transition-$\mathcal{S}$-equiloaded x-languages will be denoted by $\mathscr{L}_{\delta-SEQ-x}(\mathcal{S})$.

Similarly, a *strictly state-$\mathcal{S}$-equiloaded x-language accepted by empty memory*, and a *strictly transition-$\mathcal{S}$-equiloaded x-language accepted by empty memory* may be defined. We shall denote the family of strictly state-$\mathcal{S}$-equiloaded x-languages accepted by empty memory by $\mathscr{N}_{K-SEQ-x}(\mathcal{S})$, and the family of strictly transition-$\mathcal{S}$-equiloaded x-languages accepted by empty memory by $\mathscr{N}_{\delta-SEQ-x}(\mathcal{S})$.

In this thesis, we shall be concerned with three particular choices of $\mathcal{S}$ only. The first choice is $\mathcal{S} = \mathcal{C}$, defined for every ADA A by $\mathcal{C}(A) = \mathrm{Comp}(A)$, i.e., the set of *all* computation paths of the automaton A. The second choice is $\mathcal{S} = \mathcal{A}$, defined for every ADA A by $\mathcal{A}(A) = \mathrm{Acc}(A)$, i.e., the set of *all accepting* computation paths of the automaton A. Finally, the third choice is $\mathcal{S} = \mathcal{E}$, defined for every ADA A by $\mathcal{E}(A) = \mathrm{eAcc}(A)$, i.e., the set of *all* computation paths of the automaton A *accepting by empty memory*.

1.5 $\mathcal{S}$-Equiloadedness

In this section, we shall proceed to the definition of *$\mathcal{S}$-equiloadedness*, representing a formalization of a second, conceptually different idea of balanced use of resources. As in the case of strict $\mathcal{S}$-equiloadedness, the parameter $\mathcal{S}$ specifies a set of computation paths considered, however in a slightly different way.

In contrast with strict $\mathcal{S}$-equiloadedness, we shall not require the resources to be used approximately the same number of times in every computation path from $\mathcal{S}(A)$. Instead, we shall consider the (infinite) sequence of *finite* sets of computation paths

$$\{\mathcal{S}(A, n)\}_{n=0}^{\infty} = \{\mathcal{S}(A, 0), \mathcal{S}(A, 1), \mathcal{S}(A, 2), \ldots\},$$

add up the numbers of uses of each resource in every of these finite sets, and require the resulting numbers of uses to be approximately the same in limit for $n \to \infty$. To make this intuitive requirement formal, we shall define the concept of *equiloadedness quotient* and *equiloadedness measure*.

The presented definition is a generalization of the definition that we have presented in [25] in order to study the balanced use of transitions in DFA. It also has a relation to definitions from [27] and [28]. This relation will become clear in Chapter 2, where we shall observe that these definitions are equivalent for DFA and DFAε.

In addition to $\mathcal{S}$-equiloadedness, we shall define also the concept of *weak $\mathcal{S}$-equiloadedness* (by a relaxation of the requirements imposed on $\mathcal{S}$-equiloaded automata).

The formal definitions of an equiloadedness quotient and of an equiloadedness measure are as follows.

Definition 1.5.1 Let $\mathcal{S}$ be a function which for each pair (A, n), A being an ADA and n a nonnegative integer, returns some finite set $\mathcal{S}(A, n)$ of computation paths of A. Let A be an ADA with the set of states K, and the set of transitions D. Let us denote $S_n := \mathcal{S}(A, n)$. Then we define the equiloadedness $\mathcal{S}$-quotients as follows:

a) The *n-th state-equiloadedness $\mathcal{S}$-quotient* of the automaton A is defined by:

$$\beta_A(\mathcal{S}, n) = \frac{\min_{p \in K} \#[p, S_n] + 1}{\max_{q \in K} \#[q, S_n] + 1}.$$

b) The *n-th transition-equiloadedness $\mathcal{S}$-quotient* of the automaton A is defined by:

$$B_A(\mathcal{S}, n) = \frac{\min_{e \in D} \#[e, S_n] + 1}{\max_{f \in D} \#[f, S_n] + 1}.$$

Moreover, we define the equiloadedness $\mathcal{S}$-measures as follows:

a) The (lower) *state-equiloadedness $\mathcal{S}$-measure* of an automaton A is defined by:

$$\beta_A(\mathcal{S}) = \liminf_{n \to \infty} \beta_A(\mathcal{S}, n).$$

b) The (lower) *transition-equiloadedness $\mathcal{S}$-measure* of an automaton A is defined by:

$$B_A(\mathcal{S}) = \liminf_{n \to \infty} B_A(\mathcal{S}, n).$$

The equiloadedness measure of an ADA is clearly a real number from the closed interval $[0, 1]$, with the value 1 representing the most balanced use of resources and the value 0 representing the least balanced use of resources. This motivates our definitions of *$\mathcal{S}$-equiloadedness* and *weak $\mathcal{S}$-equiloadedness*.

Definition 1.5.2 Let $\mathcal{S}$ be a function which for each pair (A, n), A being an ADA and n a nonnegative integer, returns some finite set $\mathcal{S}(A, n)$ of computation paths of A. Let A be an ADA.

a) A is said to be (weakly) state-$\mathcal{S}$-equiloaded, if $\beta_A(\mathcal{S}) = 1$ ($\beta_A(\mathcal{S}) > 0$).

b) A is said to be (weakly) transition-$\mathcal{S}$-equiloaded, if $B_A(\mathcal{S}) = 1$ ($B_A(\mathcal{S}) > 0$).

Further, let x in $\{\text{DFA}, \text{DFA}\varepsilon, \text{DOCA}, \ldots\}$ be an abbreviation of some model of computation that is a special case of ADA. A language L is said to be a (weakly) *state-$\mathcal{S}$-equiloaded x-language*, if a (weakly) state-$\mathcal{S}$-equiloaded ADA A of type x exists, such that $L(A) = L$. Similarly, a language L is said to be a (weakly) *transition-$\mathcal{S}$-equiloaded x-language*, if a (weakly) transition-$\mathcal{S}$-equiloaded ADA A of type x exists, such that $L(A) = L$.

We shall denote the family of all (weakly) state-$\mathcal{S}$-equiloaded x-languages by the symbol $\mathscr{L}_{K-EQ-x}(\mathcal{S})$ ($\mathscr{L}_{K-WEQ-x}(\mathcal{S})$). Similarly, we shall denote the family of all (weakly) transition-$\mathcal{S}$-equiloaded x-languages by $\mathscr{L}_{\delta-EQ-x}(\mathcal{S})$ ($\mathscr{L}_{\delta-WEQ-x}(\mathcal{S})$).

Analogously, we define the families of (weakly) $\mathcal{S}$-equiloaded x-languages accepted by empty memory. The difference in the notation is in the use of the symbol $\mathscr{N}$ instead of $\mathscr{L}$.

In this thesis, we shall be concerned mainly by the following six particular choices of the parameter $\mathcal{S}$:

- $\mathcal{S}(A,n) = \mathcal{C}_{=}(A,n) = \mathrm{Comp}(A,n)$
- $\mathcal{S}(A,n) = \mathcal{A}_{=}(A,n) = \mathrm{Acc}(A,n)$
- $\mathcal{S}(A,n) = \mathcal{E}_{=}(A,n) = \mathrm{eAcc}(A,n)$
- $\mathcal{S}(A,n) = \mathcal{C}_{\leq}(A,n) = \mathrm{Comp}(A,\leq n)$
- $\mathcal{S}(A,n) = \mathcal{A}_{\leq}(A,n) = \mathrm{Acc}(A,\leq n)$
- $\mathcal{S}(A,n) = \mathcal{E}_{\leq}(A,n) = \mathrm{eAcc}(A,\leq n)$

In what follows, we shall state a lemma that provides us with an alternative formula for the computation of equiloadedness $\mathcal{S}$-measures. We shall use the lemma extensively in this thesis, since it makes the manipulation with equiloadedness $\mathcal{S}$-measures easier. For DFA and DFAε, the following lemma will allow us to numerically compute equiloadedness $\mathcal{S}$-measures.

Lemma 1.5.3 Let $\mathcal{S}$ be a function which for each pair (A,n), A being an ADA and n a nonnegative integer, returns some finite set $\mathcal{S}(A,n)$ of computation paths of A. Let A be an ADA with the set of states K, and the set of transitions D. Let us denote $S_n := \mathcal{S}(A,n)$. Then

$$\beta_A(\mathcal{S}) = \min_{(p,q)\in K^2} \liminf_{n\to\infty} \frac{\#[p,S_n]+1}{\#[q,S_n]+1}, \tag{1.1}$$

and

$$B_A(\mathcal{S}) = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\#[e,S_n]+1}{\#[f,S_n]+1}. \tag{1.2}$$

Proof. We shall prove the equation (1.1) for states, the equation (1.2) for transitions can be proved in a similar manner. First, we shall prove that the left side of (1.1) is less than or equal to the right side of (1.1). Let p', q' in K be states of the automaton A. For all n in $\mathbb{N}$, we have

$$\#[p',S_n] \geq \min_{p\in K} \#[p,S_n],$$
$$\#[q',S_n] \leq \max_{q\in K} \#[q,S_n].$$

Thus, for all n in $\mathbb{N}$, the inequality

$$\beta_A(\mathcal{S},n) = \frac{\min_{p\in K}\#[p,S_n]+1}{\max_{q\in K}\#[q,S_n]+1} \leq \frac{\#[p',S_n]+1}{\#[q',S_n]+1}$$

holds. Thus,

$$\beta_A(\mathcal{S}) = \liminf_{n\to\infty} \beta_A(\mathcal{S},n) \leq \liminf_{n\to\infty} \frac{\#[p',S_n]+1}{\#[q',S_n]+1}. \tag{1.3}$$

Since (1.3) holds for all p', q' in K, we can clearly conclude

$$\beta_A(\mathcal{S}) \leq \min_{(p,q)\in K^2} \liminf_{n\to\infty} \frac{\#[p,S_n]+1}{\#[q,S_n]+1}.$$

Now, let us prove that the left side of (1.1) is greater than or equal to the right side of (1.1). Let us denote

$$m_n := \min_{p\in K} \#[p,S_n],$$
$$M_n := \max_{q\in K} \#[q,S_n].$$

For the purpose of contradiction, let us suppose that the inequality

$$\beta_A(\mathcal{S}) = \liminf_{n\to\infty} \frac{m_n+1}{M_n+1} < \min_{(p,q)\in K^2} \liminf_{n\to\infty} \frac{\#[p,S_n]+1}{\#[q,S_n]+1} =: \ell \tag{1.4}$$

holds. Let us denote

$$\lambda := \frac{\beta_A(\mathcal{S}) + \ell}{2}.$$

From (1.4), it is possible to conclude that there is an infinite sequence $\{j_n\}_{n=0}^{\infty}$ of nonnegative integers, such that

$$\frac{m_{j_n} + 1}{M_{j_n} + 1} < \lambda$$

for all n in $\mathbb{N}$. Since the set of all pairs of states is finite, and since for all n in $\mathbb{N}$, a pair of states (p_n, q_n) in K^2 exists, such that

$$\#[p_n, S_n] = m_n, \qquad \text{and} \qquad \#[q_n, S_n] = M_n,$$

there is an infinite sequence $\{k_n\}_{n=0}^{\infty}$ of nonnegative integers, and states p', q' in K, such that

$$\frac{m_{j_{k_n}} + 1}{M_{j_{k_n}} + 1} = \frac{\#[p', S_{j_{k_n}}] + 1}{\#[q', S_{j_{k_n}}] + 1} < \lambda$$

holds for all n in $\mathbb{N}$. Thus, we have

$$\liminf_{n\to\infty} \frac{\#[p', S_n] + 1}{\#[q', S_n] + 1} \leq \lambda,$$

and, as a consequence,

$$\min_{(p,q)\in K^2} \liminf_{n\to\infty} \frac{\#[p, S_n] + 1}{\#[q, S_n] + 1} \leq \lambda,$$

which contradicts our assumption that

$$\min_{(p,q)\in K^2} \liminf_{n\to\infty} \frac{\#[p, S_n] + 1}{\#[q, S_n] + 1} = \ell.$$

The lemma is proved. □

1.6 Relations between the Families of Equiloaded Languages

In this section, we shall briefly examine some relations between the families of strictly $\mathcal{S}$-equiloaded and $\mathcal{S}$-equiloaded x-languages that hold for arbitrary x (being an abbreviation of some computational model that is a special case of ADA). These generally valid relations are rather basic. We shall prove more involved relations later in this thesis for particular models of computation.

Theorem 1.6.1 Let $\mathcal{S}$ be a function which for each pair (A, n), A being an ADA and n a nonnegative integer, returns some finite set $\mathcal{S}(A, n)$ of computation paths of the automaton A. Let x in $\{\text{DFA}, \text{DFA}\varepsilon, \text{DPDA}, \ldots\}$ be an abbreviation of some model of computation that is a special case of ADA. Then, the following inclusions hold:

1. $\mathscr{L}_{K-EQ-x}(\mathcal{S}) \subseteq \mathscr{L}_{K-WEQ-x}(\mathcal{S})$,
2. $\mathscr{L}_{\delta-EQ-x}(\mathcal{S}) \subseteq \mathscr{L}_{\delta-WEQ-x}(\mathcal{S})$,
3. $\mathscr{N}_{K-EQ-x}(\mathcal{S}) \subseteq \mathscr{N}_{K-WEQ-x}(\mathcal{S})$,
4. $\mathscr{N}_{\delta-EQ-x}(\mathcal{S}) \subseteq \mathscr{N}_{\delta-WEQ-x}(\mathcal{S})$.

(In the claims 3 and 4, it is assumed that automata of the type x have an ability to accept by empty memory.)

Proof. Let A be an ADA. If $\beta_A(\mathcal{S}) = 1$, then also $\beta_A(\mathcal{S}) > 0$. Thus, every state-$\mathcal{S}$-equiloaded ADA is also weakly state-$\mathcal{S}$-equiloaded. Similarly for transitions. □

Theorem 1.6.2 Let x in $\{\text{DFA}, \text{DFA}\varepsilon, \text{DPDA}, \ldots\}$ be an abbreviation of some model of computation that is a special case of ADA. Then, the following inclusions hold:

1. $\mathscr{L}_{K-SEQ-x}(\mathcal{C}) \subseteq \mathscr{L}_{K-EQ-x}(\mathcal{C}_=)$,
2. $\mathscr{L}_{\delta-SEQ-x}(\mathcal{C}) \subseteq \mathscr{L}_{\delta-EQ-x}(\mathcal{C}_=)$,
3. $\mathscr{L}_{K-SEQ-x}(\mathcal{A}) \subseteq \mathscr{L}_{K-EQ-x}(\mathcal{A}_=)$,
4. $\mathscr{L}_{\delta-SEQ-x}(\mathcal{A}) \subseteq \mathscr{L}_{\delta-EQ-x}(\mathcal{A}_=)$,
5. $\mathscr{N}_{K-SEQ-x}(\mathcal{C}) \subseteq \mathscr{N}_{K-EQ-x}(\mathcal{C}_=)$,
6. $\mathscr{N}_{\delta-SEQ-x}(\mathcal{C}) \subseteq \mathscr{N}_{\delta-EQ-x}(\mathcal{C}_=)$,
7. $\mathscr{N}_{K-SEQ-x}(\mathcal{E}) \subseteq \mathscr{N}_{K-EQ-x}(\mathcal{E}_=)$,
8. $\mathscr{N}_{\delta-SEQ-x}(\mathcal{E}) \subseteq \mathscr{N}_{\delta-EQ-x}(\mathcal{E}_=)$.

(It is assumed in the claims 5 – 8 that automata of the type x have an ability to accept by empty memory.)

Proof. We shall prove the theorem only for the case of transition-equiloadedness, the case of state-equiloadedness is analogous.

Let $\mathcal{S}_=$ be a function in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{E}_=\}$, and $\mathcal{S}$ be a function defined for every abstract deterministic automaton A by

$$\mathcal{S}(A) = \bigcup_{n=0}^{\infty} \mathcal{S}_=(A,n)$$

(i.e., if $\mathcal{S}_= = \mathcal{C}_=$, then $\mathcal{S} = \mathcal{C}$, and similarly for the other two choices of $\mathcal{S}_=$).

It clearly suffices to prove that if a given abstract deterministic automaton A with the set of transitions D is strictly transition-$\mathcal{S}$-equiloaded, then it is also transition-$\mathcal{S}_=$-equiloaded.

Since the number of transitions of any abstract deterministic automaton is finite, and since the identity

$$\sum_{e \in D} \#[e, \mathcal{S}_=(A,n)] = n \cdot |\mathcal{S}_=(A,n)|$$

holds, it follows from the Pigeonhole principle that

$$\max_{f \in D} \#[f, \mathcal{S}_=(A,n)] \geq \frac{n}{|D|} \cdot |\mathcal{S}_=(A,n)|. \tag{1.5}$$

Now, if the abstract deterministic automaton A is strictly transition-$\mathcal{S}$-equiloaded, the obvious inclusion $\mathcal{S}_=(A,n) \subseteq \mathcal{S}(A)$ implies

$$\begin{aligned}
\min_{e \in D} \#[e, \mathcal{S}_=(A,n)] &= \min_{e \in D} \sum_{\gamma \in \mathcal{S}_=(A,n)} \#[e,\gamma] \geq \max_{f \in D} \sum_{\gamma \in \mathcal{S}_=(A,n)} (\#[f,\gamma] - \eta) = \\
&= \left(\max_{f \in D} \sum_{\gamma \in \mathcal{S}_=(A,n)} \#[f,\gamma] \right) - \eta \cdot |\mathcal{S}_=(A,n)| = \\
&= \max_{f \in D} \#[f, \mathcal{S}_=(A,n)] - \eta \cdot |\mathcal{S}_=(A,n)|,
\end{aligned}$$

for some real constant η in $\mathbb{R}$. From this inequality, we obtain

$$\begin{aligned}
B_A(\mathcal{S}_=) &= \liminf_{n \to \infty} \frac{\min_{e \in D} \#[e, \mathcal{S}_=(A,n)] + 1}{\max_{f \in D} \#[f, \mathcal{S}_=(A,n)] + 1} \geq \\
&\geq \liminf_{n \to \infty} \frac{\max_{f \in D} \#[f, \mathcal{S}_=(A,n)] - \eta \cdot |\mathcal{S}_=(A,n)| + 1}{\max_{f \in D} \#[f, \mathcal{S}_=(A,n)] + 1} = \\
&= \liminf_{n \to \infty} \left(1 - \frac{\eta \cdot |\mathcal{S}_=(A,n)|}{\max_{f \in D} \#[f, \mathcal{S}_=(A,n)] + 1} \right) = 1,
\end{aligned}$$

if the automaton A is strictly transition-$\mathcal{S}$-equiloaded, since, by the inequality (1.5),

$$0 \leq \limsup_{n \to \infty} \frac{\eta \cdot |\mathcal{S}_=(A,n)|}{\max_{f \in D} \#[f, \mathcal{S}_=(A,n)] + 1} \leq \limsup_{n \to \infty} \frac{\eta \cdot |\mathcal{S}_=(A,n)|}{\frac{n}{|D|} \cdot |\mathcal{S}_=(A,n)| + 1} = 0,$$

i.e.,

$$\limsup_{n\to\infty} \frac{\eta \cdot |\mathcal{S}_{=}(A,n)|}{\max_{f\in D} \#[f, \mathcal{S}_{=}(A,n)] + 1} = 0.$$

Thus, we have proved that if the automaton A is strictly transition-$\mathcal{S}$-equiloaded, then the inequality $B_A(\mathcal{S}_{=}) \geq 1$ holds. However, the converse inequality holds trivially. That is, $B_A(\mathcal{S}_{=}) = 1$, and the automaton A is transition-$\mathcal{S}_{=}$-equiloaded. The theorem is proved. □

Theorem 1.6.3 Let x in $\{\text{DFA}, \text{DFA}\varepsilon, \text{DPDA}, \ldots\}$ be an abbreviation of some model of computation that is a special case of ADA. Then, the following inclusions hold:

1. $\mathscr{L}_{K-SEQ-x}(\mathcal{C}) \subseteq \mathscr{L}_{K-SEQ-x}(\mathcal{A})$,
2. $\mathscr{L}_{\delta-SEQ-x}(\mathcal{C}) \subseteq \mathscr{L}_{\delta-SEQ-x}(\mathcal{A})$,
3. $\mathscr{N}_{K-SEQ-x}(\mathcal{C}) \subseteq \mathscr{N}_{K-SEQ-x}(\mathcal{E})$.
4. $\mathscr{N}_{\delta-SEQ-x}(\mathcal{C}) \subseteq \mathscr{N}_{\delta-SEQ-x}(\mathcal{E})$.

(It is assumed in the claims 3 – 4 that automata of the type x have an ability to accept by empty memory.)

Proof. It is obvious that for all abstract deterministic automata A of the type x, the inclusions $\mathcal{C}(A) \supseteq \mathcal{A}(A)$ and $\mathcal{C}(A) \supseteq \mathcal{E}(A)$ hold. Thus, if the inequality

$$|\#[p,\gamma] - \#[q,\gamma]| \leq \eta,$$

where η in $\mathbb{R}$ is a real constant holds for every two states p, q in K and every computation path γ in $\mathcal{C}(A)$, then it holds also for every computation path γ in $\mathcal{A}(A)$, and for every computation path γ in $\mathcal{E}(A)$.

Thus, if the automaton A is strictly state-$\mathcal{C}$-equiloaded, then it is also strictly state-$\mathcal{A}$-equiloaded and strictly state-$\mathcal{E}$-equiloaded. The same property can be analogously proved also for the case of transition-equiloadedness. The inclusions from the statement of the theorem then follow as a consequence. □

1.7 Prefix-Dense Languages and Strict $\mathcal{S}$-Equiloadedness

In this section, we shall introduce a notion of a prefix-dense language and connect this notion to the theory of strict $\mathcal{S}$-equiloadedness. The theory developed in this section will serve as a useful method for proving that certain languages are not strictly $\mathcal{S}$-equiloaded.

We shall achieve this proof method as follows: first, we shall prove that languages belonging to certain families of strictly $\mathcal{S}$-equiloaded languages are always prefix-dense (this will be done independently from the model of computation, i.e., for abstract deterministic automata). However, prefix-density of the language will be defined in such a manner that this property is usually easy to disprove for a given language.

Definition 1.7.1 Let L be an arbitrary language. The language L is said to be *prefix-dense*, if a nonnegative integer constant K in $\mathbb{N}$ exists, such that for every word w in L, the following property holds: let i, j in $\mathbb{N}$, $0 \leq i \leq j \leq |w|$, $j - i \geq K$, be nonnegative integers. Then, a nonnegative integer k in $\mathbb{N}$, $i \leq k \leq j$ exists, such that the prefix $w[1 \ldots k]$ of the word w is in L.

That is, the language L is said to be prefix-dense, if a constant K exists, such that for all words w from the language L, at least one of every $K + 1$ consecutive prefixes of w is in L. In the following lemma, we shall prove an alternative description of prefix-density. This alternative description is useful, since it can be manipulated more easily.

Lemma 1.7.2 Let L be an arbitrary language. The language L is prefix-dense, if and only if a nonnegative integer constant K' in $\mathbb{N}$ exists, such that for all words w in L, the property

$$\text{Pref}_{w,K'}(i) \cap L \neq \emptyset, \qquad i = 0, \ldots, \lfloor |w|/K' \rfloor \tag{1.6}$$

holds, where $\mathrm{Pref}_{w,K'}(i)$ is a language defined by

$$\mathrm{Pref}_{w,K'}(i) = \{ w[1 \ldots k] \mid k = iK', \ldots, (i+1)K' - 1 \}$$

for $i = 0, \ldots, \lfloor |w|/K' \rfloor - 1$, and by

$$\mathrm{Pref}_{w,K'}(i) = \{ w[1 \ldots k] \mid k = iK', \ldots, |w| \}$$

for $i = \lfloor |w|/K' \rfloor$.

Proof. If the language L is prefix-dense in the sense of Definition 1.7.1, for some constant K, then it clearly satisfies also the condition stated by this lemma, for the constant $K' = K + 1$: the language $\mathrm{Pref}_{w,K'}(\lfloor |w|/K' \rfloor)$ contains the word w, and thus (1.6) is satisfied for $i = \lfloor |w|/K' \rfloor$. The languages $\mathrm{Pref}_{w,K'}(i)$, $i = 0, \ldots, \lfloor |w|/K' \rfloor - 1$, consist each of $K' = K + 1$ consecutive prefixes of the word w. Thus, since Definition 1.7.1 is satisfied by assumption, the property (1.6) holds also for $i = 0, \ldots, \lfloor |w|/K' \rfloor - 1$.

Conversely, let the language L satisfy the condition imposed by this lemma, for some constant K'. It can be easily observed that it also satisfies the condition imposed by Definition 1.7.1, for $K = 2K'$. □

Before making a link to the theory of strict $\mathcal{S}$-equiloadedness, we shall state one more-or-less trivial observation:

Lemma 1.7.3 Let L be a finite language. Then it is prefix-dense.

Proof. Let l be the length of the longest word in the language L. Then, the condition imposed in Lemma 1.7.2 is clearly satisfied for $K' = l + 1$. □

Notation 1.7.4 We shall denote the family of all prefix-dense languages by $\mathscr{L}_{prefix}$.

The observation that is of crucial importance for the theory of strict $\mathcal{S}$-equiloadedness may be stated as follows:

Theorem 1.7.5 Let x in $\{\mathrm{DFA}, \mathrm{DFA}\varepsilon, \mathrm{DPDA}, \ldots\}$ be an abbreviation of some model of computation that is a special case of abstract deterministic automata. Let $\mathcal{S}$ be a function in $\{\mathcal{C}, \mathcal{A}\}$. Then, the following inclusions hold:

1. $\mathscr{L}_{K-SEQ-x}(\mathcal{S}) \subseteq \mathscr{L}_{prefix}$,
2. $\mathscr{L}_{\delta-SEQ-x}(\mathcal{S}) \subseteq \mathscr{L}_{prefix}$.

Proof. By Theorem 1.6.3, it suffices to prove the theorem for the case $\mathcal{S} = \mathcal{A}$. The remaining case $\mathcal{S} = \mathcal{C}$ is an immediate corollary.

Let us first prove that $\mathscr{L}_{K-SEQ-x}(\mathcal{A}) \subseteq \mathscr{L}_{prefix}$. For the purpose of contradiction, let us suppose that a language L in $\mathscr{L}_{K-SEQ-x}(\mathcal{A})$ exists, such that L is not prefix-dense (i.e., not in $\mathscr{L}_{prefix}$). Let $A = (K, \Sigma, G, H, \zeta, Z, \delta, q_0, F)$ be a strictly state-$\mathcal{A}$-equiloaded abstract deterministic automaton, such that $L(A) = L$.

Since the automaton A is strictly state-$\mathcal{A}$-equiloaded, a nonnegative real constant η in $\mathbb{R}$ exists, such that for all p, q in K and all accepting computation paths γ in $\mathrm{Acc}(A)$, the property

$$|\#[p, \gamma] - \#[q, \gamma]| \leq \eta$$

holds. Moreover, by Lemma 1.7.3, if the language L is finite, it is in $\mathscr{L}_{prefix}$. Thus, we may assume that the language L is infinite. Thus, the language L is also nonempty and therefore, at least one accepting state q_F in F exists.

Since the language L is not prefix-dense, it is clear that for every s in $\mathbb{N}$, words $u_s, v_s \in \Sigma^*$ exist, such that $u_s v_s$ is in L, $|v_s| \geq s$ and that there is no nonempty prefix w of v_s, such that $u_s w$ is in L.

Let γ_s be the accepting computation path of the automaton A on the word $u_s v_s$. Let γ' be the computation path of the automaton A on the word u_s (determinism implies that γ' is a prefix of γ). The property

$$\#[q_F, \gamma'] \leq \#[q, \gamma'] + \eta \tag{1.7}$$

has to hold for the computation path γ' and for all q in K. The reason for this is as follows: clearly, this property has to hold for the empty computation path. Moreover, it has to hold for every accepting prefix of the computation path γ' (of course, such a prefix need not exist). Let us denote by γ'' the longest such prefix (if there is not any, let us define γ'' to be the empty computation path). We have

$$\#[q_F, \gamma''] \leq \#[q, \gamma''] + \eta.$$

However, we also clearly have

$$\#[q_F, \gamma'] = \#[q_F, \gamma'']$$

and

$$\#[q, \gamma'] \geq \#[q, \gamma'']$$

for all q in K. Thus, the inequality (1.7) is proved. However, since the state q_F is accepting, we also have

$$\#[q_F, \gamma] = \#[q_F, \gamma']. \tag{1.8}$$

On the other hand, it follows from the Pigeonhole principle that for at least one p in K, we have

$$\#[p, \gamma] \geq \#[p, \gamma'] + \lfloor s/|K| \rfloor . \tag{1.9}$$

Thus, by (1.7), (1.8) and (1.9) we obtain

$$|\#[q_F, \gamma] - \#[p, \gamma]| \geq \lfloor s/|K| \rfloor - \eta.$$

However, since η is a constant and $\lfloor s/|K| \rfloor$ can be made arbitrarily high, this contradicts our assumption that the abstract deterministic automaton A is strictly state-$\mathcal{S}$-equiloaded.

The proof of the inclusion $\mathscr{L}_{\delta-SEQ-x}(\mathcal{A}) \subseteq \mathscr{L}_{prefix}$ is similar. The same train of thought can be followed, with the difference that instead in the number of uses of q_F, we would be interested in the number of uses of some transition leading to q_F. The details are left to the reader. □

Example 1.7.6 The language $L_1 = \{a, b\}^*$ is clearly prefix-dense, since for every given w in L_1, every prefix of w is in L_1 as well.

However, we shall be more interested in languages that are not prefix-dense, since in that case we can also make negative statements about their strict $\mathcal{S}$-equiloadedness.

The language $L_2 = \{a^n b^n \mid n \geq 0\}$ is not prefix-dense, since for each n in $\mathbb{N}$, a counterexample $w_n = a^n b^n$ in L_2 exists, such that none of its $2n-1$ proper prefixes

$$w_n[1 \ldots 1], w_n[1 \ldots 2], \ldots, w_n[1 \ldots 2n-1]$$

is in L_2. For the similar reason, also the language $L_3 = \{a^n b \mid n \geq 1\} \cup \{\varepsilon\}$ is not prefix-dense. Thus, by Theorem 1.7.5, it follows that the languages L_2 and L_3 are not in any of the families $\mathscr{L}_{K-SEQ-x}(\mathcal{C})$, $\mathscr{L}_{K-SEQ-x}(\mathcal{A})$, $\mathscr{L}_{\delta-SEQ-x}(\mathcal{C})$, and $\mathscr{L}_{\delta-SEQ-x}(\mathcal{A})$, where x is an abbreviation of some computation model that is a special case of abstract deterministic automata.

However, if we are interested in acceptance by empty memory, prefix-density has only minor implications for strict $\mathcal{S}$-equiloadedness. In Chapter 3 (more precisely, in Examples 3.1.4 and 3.1.6), we shall show that the language L_2 is in $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$, and that L_3 is both in $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and in $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$.

Chapter 2

Deterministic Finite Automata

In the previous chapter, we have studied various aspects of several equiloadedness definitions for abstract deterministic automata. This general approach resulted in several useful observations that apply to all models of computation that can be viewed as a special case of ADA.

In this chapter, we begin the study of equiloadedness for particular models of computation and concentrate on *Deterministic Finite Automata* (with or without ε-transitions). A deterministic finite automaton is considered to be the simplest reasonable model of computation (at least among the models that are special cases of ADA) and is the only model of computation, for which the balanced use of resources has been studied up to now. In [27] and [28], some families of state-equiloaded DFA have been studied. In [25], we have initiated the study of transition-equiloaded DFA.

The definitions of equiloadedness used in [27], [28], and [25] slightly differ from those used in this thesis. However, using the terminology introduced in this thesis, we can say that in [27] and [28], families of strictly state-$\mathcal{A}$-equiloaded DFA and state-$\mathcal{A}_{=}$-equiloaded DFA have been studied[1] and in [25], we have studied the families of strictly transition-$\mathcal{A}$-equiloaded DFA, transition-$\mathcal{A}_{=}$-equiloaded DFA, and weakly transition-$\mathcal{A}_{=}$-equiloaded DFA.

In this chapter, we shall briefly restate the main results obtained in [27], [28], and [25] using the terminology and notation of this thesis (proofs will be omitted). However, the main focus will be on new results (those compose the great majority of this chapter). The definitions used in this thesis are far more general than the definitions used in [27], [28], and [25] and so it is desirable to give the known results an appropriate place in our theory. Therefore, this chapter will not consist of two separated parts, one consisting of the known results and one consisting of the new results. Instead of that, we shall be switching between the known and the new results. However, if a result is already known, a citation is always included. Therefore, it shall be relatively easy to distinguish between the known and the new results.

The new results presented later in this chapter can be roughly divided into the following categories:

- All known results have been about DFA, i.e., deterministic finite automata *without* ε-transitions. In this chapter, we shall study also DFAε, i.e., deterministic finite automata *with* ε-transitions.
- The definition of $\mathcal{S}$-equiloadedness presented in this thesis is far more general than related definitions used in [27], [28], and [25]. It can be proved that these older definitions are equivalent to $\mathcal{A}_{=}$-equiloadedness. However, in this chapter we shall study also other types of $\mathcal{S}$-equiloadedness. This involves also some new results on enumeration of basic quantities for DFA and DFAε, etc.
- A similar remark applies also to strict $\mathcal{S}$-equiloadedness.

[1] Up to now, the equivalence of the definition used in [27] and [28] and the definition of state-$\mathcal{A}_{=}$-equiloaded DFA has been an open problem. However, in this chapter we shall prove this equivalence.

- In this chapter, we shall solve an open problem concerning equivalence of certain definitions. This will lead to a unification of theories developed in [27] and [28], and in [25].
- We shall study weakly state-equiloaded DFA and DFAε that have not been studied up to now.

Before we concentrate on particular families of equiloaded DFA and DFAε, we shall show that several basic quantities used in the study of equiloaded finite automata may be computed relatively easily as solutions to certain initial value problems for systems of OΔEs (i.e., recurrences). Thus, we will be able to characterize the class of functions emerging in the numerator and the denominator of equiloadedness $\mathcal{S}$-quotients and thus to state several powerful results about equiloadedness $\mathcal{S}$-measures. Moreover, as a by-product, we will obtain a numerical algorithm for computing equiloadedness $\mathcal{S}$-measures for deterministic finite automata.

Afterwards, we shall first study strictly $\mathcal{S}$-equiloaded DFA and DFAε (Section 2.2), and subsequently $\mathcal{S}$-equiloaded (Section 2.3) DFA and DFAε for diverse choices of $\mathcal{S}$.

2.1 Enumeration of Basic Quantities for DFA and DFAε

In this section, we shall present systems of OΔEs (i.e., recurrences) that allow us to easily compute exact closed forms of several basic quantities used in the study of equiloaded DFA and DFAε, such as the number of computation paths of a given length, the number of uses of a given state or transition in computation paths of a given length, etc. These quantities form the basis of the theory of $\mathcal{S}$-equiloaded finite automata and the methods of their exact computation, presented in what follows, are crucial for the further developments of our theory.

To be more specific, we shall show that these basic quantities can be computed by solving certain initial value problems for homogeneous systems of first-order linear OΔEs with constant coefficients. Initial value problems of this kind can be solved relatively easily. For a general introduction to the topic, see, e.g., [10]. The method of solving systems of this kind is briefly reviewed in the end of this section. A brief treatment of the underlying theory, including the derivation of this method, can be found also in the appendix of our report [26].

The main idea behind the construction of systems presented in this subsection can be summarized as follows. For instance, one of the quantities we are interested in is the number of computation paths of length n. Let us slightly generalize this quantity, and let us consider the number of computation paths of length n beginning in a specified state q instead of q_0. More formally, let us denote by A_q the automaton identical to A except that its initial state is q. Then, this generalized quantity can be described as the number of computation paths of length n in the automaton A_q. Every such computation path is unambigously described by some transition leading from q and some computation path of length $n-1$ beginning in the resulting state of that transition. Thus, it is clear that the number of such computation paths can be computed as a sum through all transitions (q, c, q') in D of the numbers of computation paths of length $n-1$ in the automaton $A_{q'}$. These generalized quantities for all states of the automaton thus form a homogeneous system of first-order linear OΔEs with constant coefficients. The systems for other basic quantities can be derived in a similar manner.

As we shall observe, the matrices of the systems presented are always nonnegative, and thus can be transformed into the normal form of a reducible matrix (see, e.g., [32]). Later in this chapter, we shall make use of this fact and apply the Perron-Frobenius theory (see, e.g., [32]) to study the asymptotic properties of the basic quantities studied in this section.

Now we shall derive the systems for our basic quantities. While doing so, we shall use the following notation:

Notation 2.1.1 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε. Let q in K be a state. By A_q, we shall denote the deterministic finite automaton

$$A_q = (K, \Sigma, \delta, q, F),$$

i.e., the automaton A with the initial state replaced by q. Obviously, $A_{q_0} = A$.

Notation 2.1.2 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. Let (q_j, c, q_k) in D be a transition, and q_l in K be a state. We shall use the following notation:

$$\begin{aligned}
F_i(n) &= |\mathrm{Comp}(A_{q_i}, n)|, && i = 0, 1, \ldots, m-1,\\
f_i(n) &= |\mathrm{Acc}(A_{q_i}, n)|, && i = 0, 1, \ldots, m-1,\\
G_i(n) &= |\mathrm{Comp}(A_{q_i}, \leq n)|, && i = 0, 1, \ldots, m-1,\\
g_i(n) &= |\mathrm{Acc}(A_{q_i}, \leq n)|, && i = 0, 1, \ldots, m-1,
\end{aligned}$$

$$\begin{aligned}
T_i^{(q_j,c,q_k)}(n) &= \#[(q_j, c, q_k), \mathrm{Comp}(A_{q_i}, n)], && i = 0, 1, \ldots, m-1,\\
t_i^{(q_j,c,q_k)}(n) &= \#[(q_j, c, q_k), \mathrm{Acc}(A_{q_i}, n)], && i = 0, 1, \ldots, m-1,\\
U_i^{(q_j,c,q_k)}(n) &= \#[(q_j, c, q_k), \mathrm{Comp}(A_{q_i}, \leq n)], && i = 0, 1, \ldots, m-1,\\
u_i^{(q_j,c,q_k)}(n) &= \#[(q_j, c, q_k), \mathrm{Acc}(A_{q_i}, \leq n)], && i = 0, 1, \ldots, m-1,
\end{aligned}$$

$$\begin{aligned}
S_i^{q_l}(n) &= \#[q_l, \mathrm{Comp}(A_{q_i}, n)], && i = 0, 1, \ldots, m-1,\\
s_i^{q_l}(n) &= \#[q_l, \mathrm{Acc}(A_{q_i}, n)], && i = 0, 1, \ldots, m-1,\\
V_i^{q_l}(n) &= \#[q_l, \mathrm{Comp}(A_{q_i}, \leq n)], && i = 0, 1, \ldots, m-1,\\
v_i^{q_l}(n) &= \#[q_l, \mathrm{Acc}(A_{q_i}, \leq n)], && i = 0, 1, \ldots, m-1.
\end{aligned}$$

That is, to compute the number of all (all accepting) computation paths of length n, it suffices to compute $F_0(n)$ ($f_0(n)$), and similarly for the rest of the quantities.

Now, we are finally prepared to state theorems about the systems for computing our basic quantities. We shall be interested in the eigenvalues of system matrices, since they are of key importance in the method of solving OΔEs of this type (see the end of this section, the appendix of the report [26], or the textbook [10]).

Theorem 2.1.3 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. The functions

$$F_0, F_1, \ldots, F_{m-1}$$

are the unique solution to the initial value problem for the system of OΔEs

$$\mathbf{F}_n = \Delta \cdot \mathbf{F}_{n-1}, \qquad n \geq 1,$$

with $\mathbf{F}_n$ denoting a column vector

$$\mathbf{F}_n = (F_0(n), F_1(n), \ldots, F_{m-1}(n))^T$$

and with the initial conditions given by

$$\mathbf{F}_0 = (\underbrace{1, 1, \ldots, 1}_{m})^T.$$

Thus, the eigenvalues corresponding to this system are precisely the eigenvalues of the transition matrix Δ.

Proof. Let us first examine the trivial case $n = 0$. Clearly, for each k, there is exactly one computation path of length 0 beginning in q_k – in particular, the empty computation path. Thus, the initial conditions are indeed as in the statement of the theorem.

Now, let $n \geq 1$. Let k be fixed. We shall try to express $F_k(n)$ in terms of $F_i(n-1)$, with i in $\{0, \ldots, m-1\}$. Clearly, every computation path γ beginning in q_k must first go through some

transition e beginning in q_k. This leads to some state q_i. Then, γ follows a computation path γ' of length $n-1$, beginning in q_i. Moreover, γ is clearly unambiguously determined by e and γ'.

Thus, $F_k(n)$ can be expressed as a sum of $F_i(n-1)$ for all i, such that there is a transition beginning in q_k, and ending in q_i (and each term in the sum is weighted by the number of such transitions). Formally,

$$F_k(n) = \sum_{(q_k,c,q_i)\in D} F_i(n-1) \tag{2.1}$$

(where q_k is fixed and the sum goes through all c and i). Now, if we write down the equation (2.1) for each k in $\{0,\ldots,m-1\}$, we obtain the system (with the sums going through all c and i)

$$\begin{aligned} F_0(n) &= \sum_{(q_0,c,q_i)\in D} F_i(n-1), \\ F_1(n) &= \sum_{(q_1,c,q_i)\in D} F_i(n-1), \\ &\vdots \\ F_{m-1}(n) &= \sum_{(q_{m-1},c,q_i)\in D} F_i(n-1). \end{aligned}$$

If we write this in the matrix-vector form, we obtain exactly the system

$$\mathbf{F}_n = \Delta \cdot \mathbf{F}_{n-1}, \qquad n \geq 1.$$

Thus, the theorem is proved. □

A similar theorem may be stated also for the computation of functions enumerating the number of all accepting computation paths of a given length.

Theorem 2.1.4 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. The functions

$$f_0, f_1, \ldots, f_{m-1}$$

are the unique solution to the initial value problem for the system of OΔEs

$$\mathbf{f}_n = \Delta \cdot \mathbf{f}_{n-1}, \qquad n \geq 1,$$

with $\mathbf{f}_n$ denoting a column vector

$$(\mathbf{f}_n = f_0(n), f_1(n), \ldots, f_{m-1}(n))^T$$

and with the initial conditions given by

$$\mathbf{f}_0 = (C_0, C_1, \ldots, C_m)^T,$$

where

$$C_i = \begin{cases} 1 & \text{if } q_i \in F \\ 0 & \text{otherwise} \end{cases} \qquad i = 0, 1, \ldots, m-1.$$

The eigenvalues corresponding to this system are, again, precisely the eigenvalues of the transition matrix Δ.

Proof. For each k, if q_k is accepting, then there is exactly one accepting computation path of length 0 beginning in q_k – the empty computation path. Otherwise, there is not any. Thus, the initial conditions are indeed as in the statement of the theorem. The rest may be proved in exactly the same way as in Theorem 2.1.3. □

The following two systems may be used to enumerate the number of all (all accepting) computation paths of a length *at most* n.

Theorem 2.1.5 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. The functions

$$G_0, G_1, \ldots, G_{m-1}, F_0, F_1, \ldots, F_{m-1}$$

are the unique solution to the initial value problem for the system of OΔEs

$$\mathbf{G}_n = \begin{pmatrix} \mathbf{I}_m & \Delta \\ \mathbf{0} & \Delta \end{pmatrix} \cdot \mathbf{G}_{n-1}, \qquad n \geq 1,$$

where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix, $\mathbf{0}$ denotes the $m \times m$ zero matrix, $\mathbf{G}_n$ denotes a column vector with $2m$ entries

$$\mathbf{G}_n = (G_0(n), G_1(n), \ldots, G_{m-1}(n), F_0(n), F_1(n), \ldots, F_{m-1}(n))^T,$$

and where the initial conditions are given by

$$\mathbf{G}_0 = (\underbrace{1, 1, \ldots, 1, 1, 1, \ldots, 1}_{2m})^T.$$

Since the system matrix is an upper triangular block matrix, from its form it is obvious that if Δ has eigenvalues $\lambda_1, \ldots, \lambda_k$ with multiplicities $\alpha_1, \ldots, \alpha_k$, then the block matrix of the system has exactly these eigenvalues plus m times the eigenvalue one.

Proof. The number of all computation paths of length at most n can be clearly expressed as a number of all computation paths of length exactly n plus the number of all compuation paths of length at most $n-1$. That is, we have

$$\begin{aligned} G_0(n) &= G_0(n-1) + F_0(n) \\ G_1(n) &= G_1(n-1) + F_1(n) \\ &\vdots \\ G_{m-1}(n) &= G_{m-1}(n-1) + F_{m-1}(n) \end{aligned}$$

with initial conditions

$$G_i(0) = 1, \qquad i = 0, 1, \ldots, m-1.$$

This can be viewed either as a nonhomogeneous system with m unknown functions, or after expressing the functions $F_0, \ldots, F_{m-1}$ from the system presented in Theorem 2.1.3, as a homogeneous system with $2m$ unknown functions

$$G_0, G_1, \ldots, G_{m-1}, F_0, F_1, \ldots, F_{m-1}.$$

In the latter case, the system can be written in a (block) matrix form as

$$\mathbf{G}_n = \begin{pmatrix} \mathbf{I}_m & \Delta \\ \mathbf{0} & \Delta \end{pmatrix} \cdot \mathbf{G}_{n-1}, \qquad n \geq 1,$$

with the initial conditions given by

$$\mathbf{G}_0 = (\underbrace{1, 1, \ldots, 1, 1, 1, \ldots, 1}_{2m})^T.$$

Thus, the theorem is proved. □

Theorem 2.1.6 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. The functions

$$g_0, g_1, \ldots, g_{m-1}, f_0, f_1, \ldots, f_{m-1}$$

are the unique solution to the initial value problem for the system of OΔEs

$$\mathbf{g}_n = \begin{pmatrix} \mathbf{I}_m & \Delta \\ \mathbf{0} & \Delta \end{pmatrix} \cdot \mathbf{g}_{n-1}, \qquad n \geq 1,$$

where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix, $\mathbf{0}$ denotes the $m \times m$ zero matrix, $\mathbf{g}_n$ denotes a column vector with $2m$ entries

$$\mathbf{g}_n = (g_0(n), g_1(n), \ldots, g_{m-1}(n), f_0(n), f_1(n), \ldots, f_{m-1}(n))^T.$$

The initial conditions are given by

$$\mathbf{g}_0 = (C_0, C_1, \ldots, C_{m-1}, D_0, D_1, \ldots, D_{m-1})^T,$$

where

$$C_i = D_i = \begin{cases} 1 & \text{if } q_i \in F \\ 0 & \text{otherwise} \end{cases} \qquad i = 0, 1, \ldots, m-1.$$

Since the system matrix is an upper triangular block matrix, from its form it is obvious that if Δ has eigenvalues $\lambda_1, \ldots, \lambda_k$ with multiplicities $\alpha_1, \ldots, \alpha_k$, then the block matrix of the system has exactly these eigenvalues plus m times the eigenvalue one.

Proof. The proof is analogous as in the case of Theorem 2.1.5. □

In what follows, we shall present the systems for the computation of the number of uses of a given transition in certain computation paths.

Theorem 2.1.7 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. Let (q_j, c, q_k) in D be a transition. The functions

$$T_0^{(q_j,c,q_k)}, T_1^{(q_j,c,q_k)}, \ldots, T_{m-1}^{(q_j,c,q_k)}, F_0, F_1, \ldots, F_{m-1}$$

are the unique solution to the initial value problem for the system of OΔEs

$$\mathbf{T}_n = \begin{pmatrix} \Delta & * \\ \mathbf{0} & \Delta \end{pmatrix} \cdot \mathbf{T}_{n-1}, \qquad n \geq 1,$$

where $\mathbf{0}$ is the $m \times m$ zero matrix, $*$ is the placeholder for an arbitrary $m \times m$ matrix,[2] $\mathbf{T}_n$ denotes a column vector with $2m$ entries

$$\mathbf{T}_n = \left(T_0^{(q_j,c,q_k)}(n), T_1^{(q_j,c,q_k)}(n), \ldots, T_{m-1}^{(q_j,c,q_k)}(n), F_0(n), F_1(n), \ldots, F_{m-1}(n) \right)^T,$$

and the initial conditions are given by

$$\mathbf{T}_0 = (\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{1, 1, \ldots, 1}_{m})^T.$$

It is clear that the block matrix of the presented system has exactly the same eigenvalues as the matrix Δ, although with doubled multiplicities.

Proof. If $n = 0$, $T_i^{(q_j,c,q_k)}(0)$ is clearly 0 for $i = 0, \ldots, m-1$, since no transition is used in computation paths of length 0.

Let $n \geq 1$. Every computation path γ of length n beginning in some state q_i can be decomposed into a transition, and a computation path γ' of length $n-1$ beginning in a state determined by that transition. If $i \neq j$, then the transition (q_j, c, q_k) is not among the transitions leading from

[2] Although we could be able to exactly characterize this block, it is irrelevant for the purposes of this thesis.

q_i, and the number of uses of the transition (q_j, c, q_k) can be therefore computed as a sum of the numbers of uses of that transition in all possible computation paths γ'. If $i = j$, then the total number of uses of the transition (q_j, c, q_k) consists of this sum plus the number of computation paths γ beginning with the transition (q_j, c, q_k).

This leads us to the system (with the sums going through all c and i)

$$\begin{aligned}
T_0^{(q_j,c,q_k)}(n) &= \sum_{(q_0,c,q_i)\in D} T_i^{(q_j,c,q_k)}(n-1)\\
T_1^{(q_j,c,q_k)}(n) &= \sum_{(q_1,c,q_i)\in D} T_i^{(q_j,c,q_k)}(n-1)\\
&\vdots\\
T_{j-1}^{(q_j,c,q_k)}(n) &= \sum_{(q_{j-1},c,q_i)\in D} T_i^{(q_j,c,q_k)}(n-1)\\
T_j^{(q_j,c,q_k)}(n) &= F_k(n-1) + \sum_{(q_j,c,q_i)\in D} T_i^{(q_j,c,q_k)}(n-1)\\
T_{j+1}^{(q_j,c,q_k)}(n) &= \sum_{(q_{j+1},c,q_i)\in D} T_i^{(q_j,c,q_k)}(n-1)\\
&\vdots\\
T_{m-1}^{(q_j,c,q_k)}(n) &= \sum_{(q_{m-1},c,q_i)\in D} T_i^{(q_j,c,q_k)}(n-1)
\end{aligned}$$

with initial conditions

$$T_i^{(q_j,c,q_k)}(0) = 0 \qquad \forall i \in \{0, 1, \ldots, m-1\}.$$

This can be viewed either as a nonhomogeneous system with m unknown functions, or as a homogeneous system with $2m$ unknown functions

$$T_0^{(q_j,c,q_k)}, T_1^{(q_j,c,q_k)}, \ldots, T_{m-1}^{(q_j,c,q_k)}, F_0, F_1, \ldots, F_{m-1},$$

incorporating the system from Theorem 2.1.3. In the latter case, the system can be written in a (block) matrix form as in the statement of the theorem. Thus, the theorem is proved. □

Theorem 2.1.8 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. Let (q_j, c, q_k) in D be a transition. The functions

$$t_0^{(q_j,c,q_k)}, t_1^{(q_j,c,q_k)}, \ldots, t_{m-1}^{(q_j,c,q_k)}, f_0, f_1, \ldots, f_{m-1}$$

are the unique solution to the initial value problem for the system of OΔEs

$$\mathbf{t}_n = \begin{pmatrix} \Delta & * \\ \mathbf{0} & \Delta \end{pmatrix} \cdot \mathbf{t}_{n-1}, \qquad n \geq 1,$$

where $\mathbf{0}$ is the $m \times m$ zero matrix, $*$ is the placeholder for an arbitrary $m \times m$ matrix, $\mathbf{t}_n$ denotes a column vector with $2m$ entries

$$\mathbf{t}_n = \left(t_0^{(q_j,c,q_k)}(n), t_1^{(q_j,c,q_k)}(n), \ldots, t_{m-1}^{(q_j,c,q_k)}(n), f_0(n), f_1(n), \ldots, f_{m-1}(n) \right)^T,$$

and the initial conditions are given by

$$\mathbf{t}_0 = (\underbrace{0, 0, \ldots, 0}_{m}, C_0, C_1, \ldots, C_{m-1})^T,$$

where

$$C_i = \begin{cases} 1 & \text{if } q_i \in F \\ 0 & \text{otherwise} \end{cases} \qquad i = 0, 1, \ldots, m-1.$$

It is clear that the block matrix of the presented system has exactly the same eigenvalues as the matrix Δ, although with doubled multiplicities.

Proof. The proof is analogous to the proof of Theorem 2.1.7. □

In a similar manner, one could devise systems that enumerate the number of uses of a given transition in computation paths of length *at most n*. We shall not state these systems explicitly in this thesis, however we suppose that the reader is able to derive these systems by himself. Moreover, the systems are presented in our report [26].

The matrices and corresponding eigenvalues for the above studied systems are summarized in Table 2.1. For the rest of the basic quantities, we shall not explicitly construct systems of OΔEs (although it is certainly possible), but we shall express these quantities in terms of quantities, for which we already have systems constructed.

Quantity	Function	Matrix	Eigenvalues
$\lvert\mathrm{Comp}(A,n)\rvert$	$F_0(n)$	Δ	$\alpha_1 \times \lambda_1, \ldots, \alpha_k \times \lambda_k$
$\lvert\mathrm{Acc}(A,n)\rvert$	$f_0(n)$	Δ	$\alpha_1 \times \lambda_1, \ldots, \alpha_k \times \lambda_k$
$\lvert\mathrm{Comp}(A,\leq n)\rvert$	$G_0(n)$	$\begin{pmatrix} \mathbf{I}_m & \Delta \\ \mathbf{0} & \Delta \end{pmatrix}$	$\alpha_1 \times \lambda_1, \ldots, \alpha_k \times \lambda_k + m \times 1$
$\lvert\mathrm{Acc}(A,\leq n)\rvert$	$g_0(n)$	$\begin{pmatrix} \mathbf{I}_m & \Delta \\ \mathbf{0} & \Delta \end{pmatrix}$	$\alpha_1 \times \lambda_1, \ldots, \alpha_k \times \lambda_k + m \times 1$
$\#[(q_j,c,q_k),\mathrm{Comp}(A,n)]$	$T_0^{(q_j,c,q_k)}(n)$	$\begin{pmatrix} \Delta & * \\ \mathbf{0} & \Delta \end{pmatrix}$	$2\alpha_1 \times \lambda_1, \ldots, 2\alpha_k \times \lambda_k$
$\#[(q_j,c,q_k),\mathrm{Acc}(A,n)]$	$t_0^{(q_j,c,q_k)}(n)$	$\begin{pmatrix} \Delta & * \\ \mathbf{0} & \Delta \end{pmatrix}$	$2\alpha_1 \times \lambda_1, \ldots, 2\alpha_k \times \lambda_k$
$\#[(q_j,c,q_k),\mathrm{Comp}(A,\leq n)]$	$U_0^{(q_j,c,q_k)}(n)$	$\begin{pmatrix} \mathbf{I}_m & \Delta & * \\ \mathbf{0} & \Delta & * \\ \mathbf{0} & \mathbf{0} & \Delta \end{pmatrix}$	$2\alpha_1 \times \lambda_1, \ldots, 2\alpha_k \times \lambda_k + m \times 1$
$\#[(q_j,c,q_k),\mathrm{Acc}(A,\leq n)]$	$u_0^{(q_j,c,q_k)}(n)$	$\begin{pmatrix} \mathbf{I}_m & \Delta & * \\ \mathbf{0} & \Delta & * \\ \mathbf{0} & \mathbf{0} & \Delta \end{pmatrix}$	$2\alpha_1 \times \lambda_1, \ldots, 2\alpha_k \times \lambda_k + m \times 1$

Table 2.1: Summary of matrices and corresponding eigenvalues for systems of OΔEs presented in this section. Sets of eigenvalues and multiplicities for the specified matrices are listed in terms of eigenvalues of the transition matrix Δ, denoted $\lambda_1, \ldots, \lambda_k$, and their multiplicities, $\alpha_1, \ldots, \alpha_k$. By m, we denote the number of states of the automaton A, i.e., the transition matrix Δ is of the type $m \times m$.

Next, we shall present a theorem on the enumeration of the number of uses of a given *state* in all (or all accepting) computation paths of length n ($\leq n$). The functions for these quantities shall be expressed in terms of functions counting the number of uses of a given transition.

Theorem 2.1.9 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$. Let q_l in K be a

state. Then the following equations hold for $q_l \neq q_0$:

$$S_0^{q_l}(n) = \sum_{(q_k,c,q_l)\in D} T_0^{(q_k,c,q_l)}(n),$$
$$s_0^{q_l}(n) = \sum_{(q_k,c,q_l)\in D} t_0^{(q_k,c,q_l)}(n),$$
$$V_0^{q_l}(n) = \sum_{(q_k,c,q_l)\in D} U_0^{(q_k,c,q_l)}(n),$$
$$v_0^{q_l}(n) = \sum_{(q_k,c,q_l)\in D} u_0^{(q_k,c,q_l)}(n).$$

The following equations hold for $q_l = q_0$:

$$S_0^{q_0}(n) = F_0(n) + \sum_{(q_k,c,q_0)\in D} T_0^{(q_k,c,q_0)}(n),$$
$$s_0^{q_0}(n) = f_0(n) + \sum_{(q_k,c,q_0)\in D} t_0^{(q_k,c,q_0)}(n),$$
$$V_0^{q_0}(n) = G_0(n) + \sum_{(q_k,c,q_0)\in D} U_0^{(q_k,c,q_0)}(n),$$
$$v_0^{q_0}(n) = g_0(n) + \sum_{(q_k,c,q_0)\in D} u_0^{(q_k,c,q_0)}(n),$$

(the sums go through all q_k in K and c in $\Sigma \cup \{\varepsilon\}$).

Proof. We consider the theorem to be obvious. □

Of course, an analogous theorem holds also for $S_i^{q_l}, s_i^{q_l}, V_i^{q_l}$, and $v_i^{q_l}$, for $i = 1, \ldots, m-1$. However, we are not interested in these functions.

It is a direct corollary of the presented theorem and of the method for solving initial value problems of our kind that solutions for functions counting the number of uses of a given state have the same form (the solutions differ only in constant coefficients) as the solutions for functions counting the number of uses of a given transition. Thus, it essentially does not matter if we are enumerating the number of uses of a given state or a given transition – the solution has always the same form, and the only difference is in the values of the constant coefficients occurring in this solution.

Now, let us briefly outline the method of solving initial value problems for homogeneous systems of first-order linear OΔEs with constant coefficients (more details can be found, e.g., in [10], or in the appendix of our report [26]). A system of this kind can be always written in a matrix-vector form

$$\mathbf{x}_n = M \cdot \mathbf{x}_{n-1}, \qquad n \geq 1, \tag{2.2}$$

where M is an $m \times m$ matrix, and for every n in $\mathbb{N}$, $\mathbf{x}_n$ is a column vector with m entries

$$\mathbf{x}_n = (x_0(n), \ldots, x_{m-1}(n))^T.$$

In an initial value problem, the vector $\mathbf{x}_0$ is given explicitly as a vector of initial conditions

$$\mathbf{x}_0 = (C_0, \ldots, C_{m-1})^T.$$

In this thesis, we are not interested in solving initial value problems precisely as described above, since instead of the closed form for $\mathbf{x}_n$, we are interested in the closed form of $x_0(n)$ only. This allows us to avoid the computation of eigenvectors, or possibly generalized eigenvectors (without loosing anything). The method of finding the closed form of $x_0(n)$ can be summarized as follows:

1. Write down the system in the matrix-vector form (2.2).

2. Compute eigenvalues of the matrix M. Let us denote distinct nonzero eigenvalues of the matrix M by $\lambda_1, \ldots, \lambda_k$, and their algebraic multiplicities by $\alpha_1, \ldots, \alpha_k$. Let us denote by α the multiplicity of a zero eigenvalue.

3. The solution for $x_0(n)$ has a form

$$x_0(n) = \sum_{i=1}^{k} \sum_{j=0}^{\alpha_i - 1} c_{i,j} \cdot n^j \lambda_i^n + \sum_{j=0}^{\alpha - 1} c_{n=j} \cdot [n = j] \tag{2.3}$$

   for some constants $c_{i,j}, i = 1, \ldots, k, j = 0, \ldots, \alpha_i - 1$ and $c_{n=j}, j = 0, \ldots, \alpha - 1$ to be determined.

4. Determine the unknown constants in (2.3) by solving the system of linear equations obtained from the initial conditions.

Thus, we may conclude that we have successfully developed a method for enumeration of all basic quantities listed in the beginning of this subsection. Moreover, by Lemma 1.5.3, the computation of equiloadedness $\mathcal{S}$-measures is (at least for several most important choices of $\mathcal{S}$) reduced to the computation of a lower limit of a certain very special form. That is, the methods presented in this subsection may be turned into a numerical algorithm for computing equiloadedness $\mathcal{S}$-measures.[3]

In our report [26], one may find an extensive example of the application of the above presented method in practice, for a particular choice of a deterministic finite automaton.

2.2 Strict $\mathcal{S}$-Equiloadedness

In this section, we shall study strictly $\mathcal{S}$-equiloaded DFA and DFAε, for $\mathcal{S} = \mathcal{C}$ and $\mathcal{S} = \mathcal{A}$. Since acceptance by empty memory does not make sense for deterministic finite automata, we shall not study the case $\mathcal{S} = \mathcal{E}$ in this section.

Although, as we shall see, the families of strictly $\mathcal{C}$-equiloaded and strictly $\mathcal{A}$-equiloaded deterministic finite automata differ, we shall observe that these differences are only minor, and that the corresponding families of languages are in fact the same, both for the case of state-equiloadedness and for the case of transition-equiloadedness.

In Subsection 2.2.1, we shall start our study of strictly $\mathcal{S}$-equiloaded deterministic finite automata by proving characterizations of the families of strictly state-$\mathcal{S}$-equiloaded and strictly transition-$\mathcal{S}$-equiloaded DFA and DFAε, for $\mathcal{S} = \mathcal{C}$ and $\mathcal{S} = \mathcal{A}$. Next, in Subsection 2.2.2, we shall study the relations between the corresponding families of languages. Finally, in Subsection 2.2.3, we shall present the closure properties of these families of languages.

Some families of strictly $\mathcal{S}$-equiloaded deterministic finite automata and corresponding families of languages have already been studied. In [27] and [28], the family of strictly state-$\mathcal{A}$-equiloaded DFA and the corresponding family of languages $\mathscr{L}_{K-SEQ-DFA}(\mathcal{A})$ have been studied. Moreover, in [25], we have studied the family of strictly transition-$\mathcal{A}$-equiloaded DFA and the corresponding family $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{A})$. All of the results on these families of automata and languages, presented in this section, have been already obtained in these earlier works. However, strict $\mathcal{C}$-equiloadedness for DFA and DFAε, as well as strict $\mathcal{S}$-equiloadedness for DFAε have not been studied yet. Thus, all of the results on the corresponding families of automata and languages, presented in this section, are new.

[3] Probably the least trivial step is the computation of eigenvalues. However, eigenvalues can be numerically computed by using, e.g., the QR algorithm (for details, see, e.g., [15]).

2.2.1 Characterizations of Strict $\mathcal{S}$-Equiloadedness for $\mathcal{S} = \mathcal{C}$ and $\mathcal{S} = \mathcal{A}$

In this subsection, we shall prove the characterizations of families of strictly state-$\mathcal{S}$-equiloaded and strictly transition-$\mathcal{S}$-equiloaded DFA and DFAε, for $\mathcal{S} = \mathcal{A}$ and $\mathcal{S} = \mathcal{C}$. The characterization of strictly state-$\mathcal{A}$-equiloaded DFA is due to [27] and [28], and we have proved the characterization of strictly transition-$\mathcal{A}$-equiloaded DFA in [25]. The remaining characterizations are new. However, we shall state all of these characterizations in two theorems.

Theorem 2.2.1 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε.

a) A is strictly state-$\mathcal{C}$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle, or is a directed multicycle through all states.

b) A is strictly state-$\mathcal{A}$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle from which some accepting state is reachable,[4] or is a directed multicycle through all states.

Proof. We shall prove only the first statement, the proof of the second statement is analogous. Let us first prove the left-to-right implication. Let the automaton A be strictly state-$\mathcal{C}$-equiloaded, i.e., let the inequality

$$|\#[p, \gamma] - \#[q, \gamma]| \leq k \tag{2.4}$$

hold for some k in $\mathbb{N}$, all computation paths γ in $\mathrm{Comp}(A)$ and all states p, q in K. For the purpose of contradiction, let us suppose that the graphical representation contains a reachable directed cycle, and at the same time, is not a directed multicycle through all states.

This clearly implies that states p, q in K exist, such that for some words u, v in Σ^*,

$$(q_0, uv) \vdash^* (p, v) \vdash^+ (p, \varepsilon),$$

where the state q is not used in the computation

$$(p, v) \vdash^+ (p, \varepsilon)$$

and is used at most once in the computation

$$(q_0, uv) \vdash^* (p, v).$$

Now, let us consider the word uv^{k+2} and let us denote the corresponding computation path by γ. Then it is clear that

$$|\#[p, \gamma] - \#[q, \gamma]| \geq k + 1,$$

which contradicts (2.4).

Now, let us prove the right-to-left implication. If the graphical representation of the automaton A does not contain any reachable directed cycle, then each state of the automaton A can be used at most once in a given computation path. That is,

$$0 \leq \#[q, \gamma] \leq 1$$

for all computation paths γ and each state q in K. Thus, for all states p, q in K and all computation paths γ,

$$|\#[p, \gamma] - \#[q, \gamma]| \leq 1,$$

and the automaton A is strictly state-$\mathcal{C}$-equiloaded.

Now, let us suppose that the graphical representation of the automaton A is a directed multicycle through all states. Then it can be clearly seen that for all states p, q in K and all computation paths γ,

$$|\#[p, \gamma] - \#[q, \gamma]| \leq 1.$$

Thus, the automaton A is strictly state-$\mathcal{C}$-equiloaded. □

[4] This means that the accepted language $L(A)$ is finite, and at the same time, there is no state q in K, such that $(q, w) \vdash^+ (q, w) \vdash^* (q_F, \varepsilon)$ for some w in Σ^* and q_F in F (i.e., there is no ε-cycle from which an accepting state is reachable). Thus, for DFA, this condition is equivalent to the finiteness of the accepted language $L(A)$.

Theorem 2.2.2 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with connected graphical representation.[5]

a) A is strictly transition-$\mathcal{C}$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle, or is a directed cycle through all states.

b) A is strictly transition-$\mathcal{A}$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle from which some accepting state is reachable, or is a directed cycle through all states.

Proof. The proof of this theorem is analogous to the proof of Theorem 2.2.1. □

Finally in this subsection, we shall state two theorems and one corollary concerning strictly $\mathcal{S}$-equiloaded DFA (i.e., deterministic finite automata without ε-transitions) that have been proved for $\mathcal{S} = \mathcal{A}$ in [27], [28] (Theorem 2.2.3), and [25] (Theorem 2.2.4 and Corollary 2.2.5). However, since the proof for the case $\mathcal{S} = \mathcal{C}$ is analogous to the proof for the case $\mathcal{S} = \mathcal{A}$, we shall omit proofs of these statements.

Theorem 2.2.3 Let L in $\mathscr{R}$ be a regular language, and $\mathcal{S}$ be in $\{\mathcal{C}, \mathcal{A}\}$. The language L is a strictly state-$\mathcal{S}$-equiloaded DFA-language, if and only if the minimal DFA accepting L is strictly state-$\mathcal{S}$-equiloaded.

Theorem 2.2.4 Let L in $\mathscr{R}$, $L \subseteq \Sigma^*$, be a regular language, and $\mathcal{S}$ be in $\{\mathcal{C}, \mathcal{A}\}$. The language L is a strictly transition-$\mathcal{S}$-equiloaded DFA-language, if and only if L is finite, or if

$$L = \{u_1 u_2 \dots u_n v\}^* \{u_1, u_1 u_2, \dots, u_1 u_2 \dots u_n\}$$

for some fixed n in $\mathbb{N}$, $u_1, u_2, \dots, u_n$ in Σ^* and v in Σ^+.

Corollary 2.2.5 Let $\mathcal{S}$ be in $\{\mathcal{C}, \mathcal{A}\}$, and L in $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{S})$ be an infinite language. Let u and v be words in L, such that $|u| \leq |v|$. Then u is a prefix of v.

2.2.2 Relations between the Families of Strictly $\mathcal{S}$-Equiloaded Languages

In this subsection, we shall prove several relations between the families of strictly $\mathcal{S}$-equiloaded DFA-languages and DFAε-languages. We shall observe that some of these families are in fact the same.

Theorem 2.2.6 The following identities hold:

1. $\mathscr{L}_{K-SEQ-DFA}(\mathcal{C}) = \mathscr{L}_{K-SEQ-DFA}(\mathcal{A}) =: \mathscr{L}_{K-SEQ-DFA}$,
2. $\mathscr{L}_{K-SEQ-DFA\varepsilon}(\mathcal{C}) = \mathscr{L}_{K-SEQ-DFA\varepsilon}(\mathcal{A}) =: \mathscr{L}_{K-SEQ-DFA\varepsilon}$,
3. $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{C}) = \mathscr{L}_{\delta-SEQ-DFA}(\mathcal{A}) =: \mathscr{L}_{\delta-SEQ-DFA}$,
4. $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}(\mathcal{C}) = \mathscr{L}_{\delta-SEQ-DFA\varepsilon}(\mathcal{A}) =: \mathscr{L}_{\delta-SEQ-DFA\varepsilon}$.

Proof. The theorem is a direct consequence of Theorem 2.2.1, Theorem 2.2.2, and the obvious fact that to each deterministic finite automaton A (with or without ε-transitions) without a reachable directed cycle in the graphical representation, from which some accepting state is reachable, an equivalent deterministic finite automaton A' exists, such that there is not any reachable directed cycle in its graphical representation: it clearly suffices to delete all states of the automaton A that

[5] This is only a minor technical assumption. If this assumption had been omitted, the rest of the characterization would have not been true for automata that, aside a connected component containing the initial state, have some number of other connected components, each consisting of one isolated state. However, to every deterministic finite automaton it is easy to find an equivalent automaton with connected graphical representation – it suffices to delete all connected components not containing the initial state.

belong to some reachable directed cycle – since there is not any accepting state reachable from these states in A, the resulting automaton A' would be clearly equivalent to A. □

Thus, we have proved that, in terms of families of languages, strict $\mathcal{C}$-equiloadedness and strict $\mathcal{A}$-equiloadedness are equivalent for DFA and DFAε. However, let us note that these concepts are not equivalent in terms of families of automata. It is obviously possible to construct a deterministic finite automaton that is strictly $\mathcal{A}$-equiloaded, but not strictly $\mathcal{C}$-equiloaded (both for states and for transitions). The family of strictly $\mathcal{A}$-equiloaded DFA (DFAε) is a proper superset of the family of strictly $\mathcal{C}$-equiloaded DFA (DFAε).

From this observation, it is also possible to conclude that, in terms of families of languages, strict $\mathcal{C}$-equiloadedness and strict $\mathcal{A}$-equiloadedness are not equivalent for general families of automata that can be viewed as a special case of ADA. It suffices to consider a family x of automata consisting of a single deterministic finite automaton that is strictly state-$\mathcal{A}$-equiloaded (strictly transition-$\mathcal{A}$-equiloaded), but not strictly state-$\mathcal{C}$-equiloaded (strictly transition-$\mathcal{C}$-equiloaded). Then, the family of languages $\mathscr{L}_{K-SEQ-x}(\mathcal{A})$ ($\mathscr{L}_{\delta-SEQ-x}(\mathcal{A})$) consists of one single regular language, while the family $\mathscr{L}_{K-SEQ-x}(\mathcal{C})$ ($\mathscr{L}_{\delta-SEQ-x}(\mathcal{C})$) is empty.

Remark 2.2.7 Henceforth, in the context of Theorem 2.2.6, we shall call strictly state-$\mathcal{S}$-equiloaded (strictly transition-$\mathcal{S}$-equiloaded) DFA-languages (DFAε-languages), for $\mathcal{S} = \mathcal{A}$ or $\mathcal{S} = \mathcal{C}$, strictly state-equiloaded (strictly transition-equiloaded) DFA-languages (DFAε-languages).

Theorem 2.2.8 The following strict inclusions hold:

1. $\mathscr{L}_{K-SEQ-DFA} \subsetneq \mathscr{L}_{K-SEQ-DFA\varepsilon}$,
2. $\mathscr{L}_{\delta-SEQ-DFA} \subsetneq \mathscr{L}_{\delta-SEQ-DFA\varepsilon}$.

Proof. The claims $\mathscr{L}_{K-SEQ-DFA} \subseteq \mathscr{L}_{K-SEQ-DFA\varepsilon}$ and $\mathscr{L}_{\delta-SEQ-DFA} \subseteq \mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ are obvious, since every DFA is also a DFAε.

We shall prove that these inclusions are proper. Let us consider the language $L = \{a\}^+$. We shall construct a DFAε $A = (K, \Sigma, \delta, q_0, F)$, such that $L(A) = L$. Let us define the automaton A as follows: $K = \{q_0, q_1\}$, $\Sigma = \{a\}$, $F = \{q_1\}$, and

$$\delta(q_0, a) = q_1,$$
$$\delta(q_1, \varepsilon) = q_0.$$

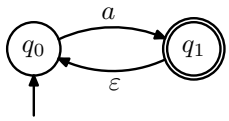


Figure 2.1: The deterministic finite automaton A with ε-transitions, accepting the language $L = \{a\}^+$.

The automaton A is strictly state-$\mathcal{S}$-equiloaded for both $\mathcal{S} = \mathcal{C}$ and $\mathcal{S} = \mathcal{A}$ (however, the equiloadedness of the automaton for one of these $\mathcal{S}$ would suffice), since its graphical representation is a directed multicycle through all states. Moreover, the graphical representation of the automaton A is in fact a directed cycle, and thus the automaton is also strictly transition-$\mathcal{S}$-equiloaded for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$. The claim $L(A) = L$ is considered to be obvious. Thus, L is both in $\mathscr{L}_{K-SEQ-DFA\varepsilon}$ and in $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$.

However, by applying Theorem 2.2.3, it is easy to prove that L is neither in $\mathscr{L}_{K-SEQ-DFA}$, nor in $\mathscr{L}_{\delta-SEQ-DFA}$ (the minimal DFA accepting L does not satisfy any of the presented characterizations of strict $\mathcal{S}$-equiloadedness). The theorem is proved. □

Theorem 2.2.9 The following strict inclusions hold:

1. $\mathscr{L}_{\delta-SEQ-DFA} \subsetneq \mathscr{L}_{K-SEQ-DFA}$,
2. $\mathscr{L}_{\delta-SEQ-DFA\varepsilon} \subsetneq \mathscr{L}_{K-SEQ-DFA\varepsilon}$.

Proof. The correctness of the inclusions is obvious, since every directed cycle is at the same time a directed multicycle (and thus, by the characterizations given in Theorem 2.2.1 and in Theorem 2.2.2, every strictly transition-$\mathcal{S}$-equiloaded DFA(ε) is strictly state-$\mathcal{S}$-equiloaded as well).

Let us prove that these inclusions are proper and consider the language $L = \{a, b\}^*$. It is obvious that L is in $\mathscr{L}_{K-SEQ-DFA}$ (since it can be accepted by a DFA consisting of one single state), and thus, by Theorem 2.2.8, also in $\mathscr{L}_{K-SEQ-DFA}$. We shall prove that L is not in $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ (and, by Theorem 2.2.8, neither in $\mathscr{L}_{\delta-SEQ-DFA}$).

For the purpose of contradiction, let us suppose that a strictly transition-$\mathcal{A}$-equiloaded DFAε $A = (K, \Sigma, \delta, q_0, F)$ exists, such that $L(A) = L$. Since the language L is infinite, the number of accepting computation paths of the automaton A has to be infinite as well. Thus, according to the characterization from Theorem 2.2.2, the graphical representation of the automaton A is a directed cycle through all states. Moreover, F is a nonempty set.

Let q in K be the first state after q_0 in the direction of the directed cycle, such that there is a transition on a character from Σ (that is, not an ε-transition), leading from q (such a state has to exist, since the language $L(A)$ is nonempty). Since the graphical representation of the automaton A is a directed cycle through all states, there has to be *exactly one* transition leading from q. But if this transition is on a, then every word from $L(A)$ has to begin with a, and if this transition is on b, then every word from $L(A)$ has to begin with b. In both cases, this is a contradiction with the assumption $L(A) = L$. □

2.2.3 Closure Properties

In this subsection, we shall state the closure properties of the families of strictly $\mathcal{S}$-equiloaded DFA-languages and DFAε-languages. However, in order to keep the length of this thesis reasonable, we shall omit proofs. The proofs of the closure properties of the families of languages that have already been examined, can be found in [27], [28], and [25]. The proofs of the closure properties of the families of languages that have not been examined yet, can be found in our report [26] (the extended version of this thesis).

Theorem 2.2.10 The closure properties of the families of strictly $\mathcal{S}$-equiloaded languages are as follows:

1. The family $\mathscr{L}_{K-SEQ-DFA}$ is closed under intersection and is not closed under concatenation, union, complementation, Kleene star, Kleene plus, reversal, homomorphism, and inverse homomorphism.
2. The family $\mathscr{L}_{K-SEQ-DFA\varepsilon}$ is closed under intersection and is not closed under concatenation, union, complementation, Kleene star, Kleene plus, reversal, homomorphism, and inverse homomorphism.
3. The family $\mathscr{L}_{\delta-SEQ-DFA}$ is closed under intersection and is not closed under concatenation, union, complementation, Kleene star, Kleene plus, reversal, homomorphism, and inverse homomorphism.
4. The family $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ is closed under intersection and homomorphism, and is not closed under concatenation, union, complementation, Kleene star, Kleene plus, reversal, and inverse homomorphism.

2.3 $\mathcal{S}$-Equiloadedness

In this section, we shall study $\mathcal{S}$-equiloaded DFA and DFAε. However, in most results presented in this section, we shall not be concerned with $\mathcal{S}$-equiloadedness for general $\mathcal{S}$, but with several special cases of $\mathcal{S}$. The most important choices of $\mathcal{S}$ are $\mathcal{C}_=, \mathcal{C}_\leq, \mathcal{A}_=$, and $\mathcal{A}_\leq$.

Some of the families of $\mathcal{S}$-equiloaded automata and languages have already been studied (although the terminology and definitions have been slightly different, since the concept of $\mathcal{S}$-equiloadedness is introduced in this thesis as a generalization of the older definitions).

In [27] and [28], state-$\mathcal{A}_=$-equiloaded deterministic finite automata without ε-transitions have been studied. However, the definition used in [27] and [28] is conceptually different from the definition used in this thesis and, up to now, the equivalence of both definitions has been an open problem. This open problem is solved in Subsection 2.3.1. Moreover, in Subsection 2.3.1, analogous equivalence theorems are proved also for $\mathcal{S} = \mathcal{C}_=$, and for deterministic finite automata with ε-transitions.

In [25], we have studied transition-$\mathcal{A}_=$-equiloaded deterministic finite automata without ε-transitions. We have used the definition from [25] as a basis for the more general definition of $\mathcal{S}$-equiloadedness used in this thesis, and can therefore be viewed as its special case. Thus, there is no need to prove the equivalence. However, in [25], we have proved the equivalence of the alternative definition of transition-$\mathcal{A}_=$-equiloadedness analogous to the definitions used in [27] and [28] for state-equiloadedness and, in Subsection 2.3.1 of this thesis, we shall prove that these alternative definitions are equivalent to transition-$\mathcal{S}$-equiloadedness also for several other choices of $\mathcal{S}$, and for DFAε.

In addition to alternative definitions, we shall also study some other aspects of $\mathcal{S}$-equiloadedness in this section. In [25], we have presented the characterization of weakly transition-$\mathcal{A}_=$-equiloaded DFA. In Subsection 2.3.5, we shall generalize this result to several other choices of $\mathcal{S}$ and to DFAε. Moreover, we shall prove also the characterization of weakly state-$\mathcal{C}_=$-equiloaded DFA and DFAε. Subsequently, we shall be interested in the relations between corresponding families of languages. Finally, in Subsection 2.3.7, we shall examine the closure properties of families of languages corresponding to $\mathcal{S}$-equiloaded DFA.

However, before we begin the study of $\mathcal{S}$-equiloaded finite automata, let us state a well-known number-theoretic lemma that we shall use extensively in our studies. The proof of the lemma can be found, e.g., in [3]. The same proof can be found also in the appendix of our report [26].

Lemma 2.3.1 Let S be a set of positive integers, satisfying the following two conditions:

(i) The greatest common divisor of the elements of S is d.

(ii) The set S is closed under addition.

Then a positive integer n_0 exists, such that for all positive integers $n \geq n_0$, such that d divides n, n is in S.

2.3.1 Alternative Definitions of $\mathcal{S}$-Equiloadedness for $\mathcal{S} = \mathcal{C}_=$ and $\mathcal{S} = \mathcal{A}_=$

In this subsection, we shall prove a theorem providing alternative definitions of state-$\mathcal{S}$-equiloadedness and transition-$\mathcal{S}$-equiloadedness for two most important choices of $\mathcal{S}$ – for $\mathcal{C}_=$, and $\mathcal{A}_=$. The basic definitions of the concept of $\mathcal{S}$-equiloadedness via the measure of $\mathcal{S}$-equiloadedness, as presented in Section 1.5, are a generalization of the definition of transition-$\mathcal{A}_=$-equiloaded DFA that we have used in [25]. The alternative definitions of $\mathcal{S}$-equiloadedness, provided by Theorem 2.3.5 presented in this subsection, are based on the definitions of state-$\mathcal{A}_=$-equiloaded DFA used in [27] and [28].

However, before we present these alternative definitions, we shall prove three lemmas that we shall use in the proof of Theorem 2.3.5. First, in Lemma 2.3.2, we shall observe that the asymptotic growth of the quantity representing the number of computation paths of length n in a strongly connected automaton is determined by the Perron-Frobenius eigenvalue of the transition matrix.

Lemma 2.3.2 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation, and Δ be the transition matrix of the automaton A. Let ρ be the Perron-Frobenius eigenvalue of the transition matrix Δ. Then

$$F_0(n) = |\mathrm{Comp}(A, n)| = \Theta(\rho^n).$$

Proof. Since the graphical representation of the automaton A is strongly connected, the transition matrix Δ is either an irreducible matrix, or the 1×1 null matrix. For the case of the 1×1 null matrix, the statement of the lemma is clearly true for $\rho = 0$ (i.e., the 1×1 null matrix can be assumed to have a zero Perron-Frobenius eigenvalue). In the rest of the proof, we shall therefore assume that the matrix Δ is irreducible.

By the Perron-Frobenius theorem, the transition matrix Δ has the Perron-Frobenius eigenvalue, i.e., the statement of the lemma makes sense. Moreover, the eigenvalues of Δ are complex numbers

$$\rho, \rho \cdot e^{2\pi i/p}, \ldots, \rho \cdot e^{2\pi i(p-1)/p}, \lambda_1, \ldots, \lambda_k,$$

where p in $\mathbb{N}^+$ is a positive integer, k in $\mathbb{N}$ is a nonnegative integer and

$$|\lambda_j| < \rho,$$

for $j = 1, \ldots, k$. Moreover, all eigenvalues of the absolute value ρ are simple, i.e., of algebraic multiplicity 1. Let us denote the algebraic multiplicities of $\lambda_1, \ldots, \lambda_k$ by $\alpha_1, \ldots, \alpha_k$.

Since the graphical representation of the automaton A is strongly connected with at least one transition (a strongly connected digraph does not have any edge only if its adjacency matrix is the 1×1 null matrix), it is clear that $F_0(n)$ is nonzero for all n. Thus, by the results obtained in Section 2.1, the transition matrix Δ has at least one nonzero eigenvalue (otherwise, every possible solution for $F_0(n)$ would be nonzero only for finite number of n) and, for n greater than some n_0 in $\mathbb{N}$, the property

$$\begin{aligned} F_0(n) &= \sum_{j=0}^{p-1} c_j \cdot \rho^n \cdot e^{2\pi i n j/p} + \sum_{j=1}^{k} \sum_{h=0}^{\alpha_j - 1} c_{j,h} \cdot n^h \cdot \lambda_j^n = \\ &= \left(\sum_{j=0}^{p-1} c_j \cdot e^{2\pi i n j/p} \right) \cdot \rho^n + \sum_{j=1}^{k} \sum_{h=0}^{\alpha_j - 1} c_{j,h} \cdot n^h \cdot \lambda_j^n \end{aligned} \tag{2.5}$$

holds (terms corresponding to zero eigenvalues have been omitted, since they have effect on the value of $F_0(n)$ only for the finite number of n) for some complex constants c_j, $j = 0, \ldots, p-1$ and $c_{j,h}$, $j = 1, \ldots, k$, $h = 0, \ldots, \alpha_j - 1$.

The function (of the variable n)

$$\sum_{j=0}^{p-1} c_j \cdot e^{2\pi i n j/p}, \tag{2.6}$$

arising in (2.5), is clearly periodic with period p. We shall prove that the value of the function (2.6) is nonzero for all n in $\mathbb{N}$.

By the results obtained in Section 2.1,

$$\mathbf{F}_n = \Delta^n \cdot \mathbf{F}_0,$$

with

$$\mathbf{F}_n = (F_0(n), F_1(n), \ldots, F_{m-1}(n))^T$$

and

$$\mathbf{F}_0 = (\underbrace{1, 1, \ldots, 1}_{m})^T,$$

where $m = |K|$. Let us denote by E^m the *standard* $(m-1)$*-simplex*, i.e., the set

$$E^m = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m \;\middle|\; x_1 \geq 0, \ldots, x_m \geq 0, \sum_{i=1}^{m} x_i = 1 \right\}.$$

Now, let $\mathbf{x}_\rho$ in E^m be the Perron-Frobenius eigenvector of the matrix Δ, corresponding to the Perron-Frobenius eigenvalue ρ. Since $\mathbf{x}_\rho$ is in E^m,

$$\mathbf{F}_0 \geq \mathbf{x}_\rho.$$

Thus,

$$\mathbf{F}_n = \Delta^n \cdot \mathbf{F}_0 \geq \Delta^n \cdot \mathbf{x}_\rho = \rho^n \cdot \mathbf{x}_\rho.$$

Since the Perron-Frobenius eigenvector $\mathbf{x}_\rho$ is always positive, we may conclude that a positive real constant Q in $\mathbb{R}^+$ exists, such that

$$F_0(n) \geq Q \cdot \rho^n. \tag{2.7}$$

Now, for the purpose of contradiction, let us suppose that (2.6) is zero for some n in $\mathbb{N}$. Then, the periodicity implies that (2.6) is zero for infinitely many n in $\mathbb{N}$. However, this clearly contradicts (2.7). Thus, the value of the function (2.6) is nonzero for all n in $\mathbb{N}$.

Moreover, the function (2.6) has to be real. Otherwise, for some j in $\{0, \ldots, p-1\}$, the function would have to attain a nonreal complex value $z = a + bi$, $b \neq 0$, for all $n \cdot p + j$, n in $\mathbb{N}$. However, for n greater than some n_0 in $\mathbb{N}$, clearly

$$\left| \sum_{j=1}^{k} \sum_{h=0}^{\alpha_j - 1} c_{j,h} \cdot n^h \cdot \lambda_j^n \right| < b \cdot \rho^n$$

and thus, the function $F_0(n)$ would have to attain nonreal complex values for n greater than n_0. However, $F_0(n)$ is clearly a real function. Thus, the function (2.6) has to be real as well.

Further, we shall prove that the function (2.6) is not only real, but in fact positive. We have already proved that the value of the function (2.6) at given n is real and nonzero. For the purpose of contradiction, let us suppose that it attains a negative value for some n. Then, by the periodicity of the function (2.6), it follows that (2.6) attains a negative value for infinitely many n. However, by applying a similar argument as above, it can be easily seen that this implies that also the function $F_0(n)$ attains at least one negative value. But this is a contradiction, since the function $F_0(n)$ is clearly nonnegative.

Thus, we have proved that (2.6) is a positive function periodic with period p. This implies that it has to be bounded from below and from above by a positive constant. This clearly implies

$$F_0(n) = |\mathrm{Comp}(A, n)| = \Theta(\rho^n).$$

Thus, the lemma is proved. □

Next, in Lemma 2.3.3, we shall prove that a similar property as in Lemma 2.3.2 holds also for the number of all *accepting* computation paths of a given length. The only difference is that in this case, lengths n, such that there is not any accepting computation of length n, are a serious problem. However, if we omit these lengths from our consideration, the property holds.

Lemma 2.3.3 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation, and Δ be the transition matrix of the automaton A. Let ρ be the Perron-Frobenius eigenvalue of the transition matrix Δ. Let F be nonempty and $\{n_k\}_{k=0}^{\infty}$ be the increasing sequence of all nonnegative integers n_k, such that at least one accepting computation path of length n_k exists in the automaton A. Then

$$f_0(n_k) = |\mathrm{Acc}(A, n_k)| = \Theta(\rho^{n_k}).$$

Proof. The statement of the lemma is obviously true in the case when Δ is the 1×1 null matrix. Thus, let us suppose that Δ is irreducible. Then $F_0(n)$ is nonzero for all n in $\mathbb{N}$ and thus, by Lemma 2.3.2, the Perron-Frobenius eigenvalue ρ of the matrix Δ is nonzero (this can be of course easily proved also without the use of Lemma 2.3.2).

Let us denote by d the greatest common divisor of lengths of the closed walks in the graphical representation of the automaton A, with the beginning and end in the vertex corresponding to the initial state q_0. Let q in F be an accepting state, and r in $\mathbb{N}$ be a nonnegative integer, such that

$$(q_0, w) \vdash^r (q, \varepsilon)$$

for some w in Σ^*.

Let S be a set of nonnegative integers s, such that a word w in Σ^* exists, such that

$$(q_0, w) \vdash^s (q_0, \varepsilon).$$

Clearly, the greatest common divisor of elements of S is d, and the set S is closed under addition. Thus, by Lemma 2.3.1, a nonnegative integer n_0 in $\mathbb{N}$ exists, such that for all n in $\mathbb{N}$, $n \geq n_0$, such that d divides n, n is in S. Let n_1 be the smallest nonnegative integer greater than or equal to n_0, such that d divides n_1.

Let k be in $\mathbb{N}$. By Lemma 2.3.2,

$$\begin{aligned} F_0(n_k - (|K| - 1) - n_1 - r) = |\mathrm{Comp}(A, n_k - (|K| - 1) - n_1 - r)| = \Theta(\rho^{n_k - (|K|-1) - n_1 - r}) = \\ = \Theta(\rho^{n_k}). \end{aligned}$$

Now, since the graphical representation of the automaton A is strongly connected, each state is reachable from each other state in at most $|K| - 1$ steps. Thus, every computation path of length $n_k - (|K| - 1) - n_1 - r$ can be prolonged to a computation path of length at most $n_k - n_1 - r$, ending in state q_0. Moreover, if two computation paths are distinct, then every two computation paths obtained by prolonging these two computation paths are distinct as well. Thus, if we denote by $\varphi(n_k - n_1 - r)$ the number of maximal[6] computation paths of length at most $n_k - n_1 - r$ ending in q_0, we have

$$\varphi(n_k - n_1 - r) \geq F_0(n_k - (|K| - 1) - n_1 - r) = \Theta(\rho^{n_k}),$$

i.e.,

$$\varphi(n_k - n_1 - r) = \Omega(\rho^{n_k}).$$

Clearly, all of these $\varphi(n_k - n_1 - r)$ computation paths are of length divisible by d. Thus, by the structure of the set S, all of these $\varphi(n_k - n_1 - r)$ can be prolonged to a computation path of length $n_k - r$ ending in q_0. Since the computation paths involved are maximal, the prolonged computation paths are all distinct. Thus, if we denote by $\psi(n_k - r)$ the number of computation paths of length exactly $n_k - r$ ending in q_0, we have

$$\psi(n_k - r) \geq \varphi(n_k - n_1 - r) = \Omega(\rho^{n_k}),$$

i.e.,

$$\psi(n_k - r) = \Omega(\rho^{n_k}).$$

However, since the accepting state q is reachable from q_0 in r steps, this implies

$$f_0(n_k) = |\mathrm{Acc}(A, n_k)| \geq \psi(n_k - r) = \Omega(\rho^{n_k}),$$

i.e.,

$$f_0(n_k) = |\mathrm{Acc}(A, n_k)| = \Omega(\rho^{n_k}).$$

[6] In the sense that a computation path cannot be prolonged to another computation path satisfying the property that its length is at most $n_k - n_1 - r$ and it ends in q_0.

However, on the other hand we have

$$f_0(n_k) = |\mathrm{Acc}(A, n_k)| \leq |\mathrm{Comp}(A, n_k)| = F_0(n_k) = \Theta(\rho^{n_k}),$$

i.e.,

$$f_0(n_k) = |\mathrm{Acc}(A, n_k)| = O(\rho^{n_k}).$$

Thus,

$$f_0(n_k) = |\mathrm{Acc}(A, n_k)| = \Theta(\rho^{n_k})$$

and the lemma is proved. □

Finally, in Lemma 2.3.4, we shall state one easy observable yet useful property that we shall use later in our study.

Lemma 2.3.4 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε, and Δ be the transition matrix of the automaton A. The transition matrix Δ has a nonzero eigenvalue if and only if the graphical representation of the automaton A contains at least one directed cycle.

Proof. Without loss of generality, let us suppose that the matrix Δ is in the normal form of a reducible matrix. The spectrum of the matrix Δ is the union of spectra of its diagonal blocks (corresponding to strongly connected components of the graphical representation of the automaton A) that are either irreducible or 1×1 null matrices.

If the graphical representation of the automaton A contains at least one directed cycle, then there is at least one strongly connected component with at least one directed cycle. Then, the number of computation paths in this strongly connected component (with arbitrary initial state) of length n has to be nonzero for all n in $\mathbb{N}$. Thus, by Lemma 2.3.2, the corresponding diagonal block has to have at least one nonzero eigenvalue (the Perron-Frobenius eigenvalue of that block).

If the graphical representation does not contain any directed cycle, then each state forms one strongly connected component. The number of computation paths in all strongly connected components is thus 1 for the computation paths of length 0 and 0 otherwise. Thus, by Lemma 2.3.2, the eigenvalue of each diagonal 1×1 block is 0 (that is, all diagonal blocks are null), and the matrix Δ does not have any nonzero eigenvalue. □

Now we are prepared to prove the key result of this subsection, providing an alternative definition of $\mathcal{S}$-equiloadedness, for $\mathcal{S} = \mathcal{C}_=$ and $\mathcal{S} = \mathcal{A}_=$. The following theorem implies that a definition of DFA with balanced use of states in accepting computations, used in [27] and [28], is equivalent to our definition of state-$\mathcal{A}_=$-equiloadedness, and that a similar property holds also for some other types of $\mathcal{S}$-equiloadedness defined in this thesis.

Theorem 2.3.5 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε. Let $\mathcal{S}$ be a function in $\{\mathcal{C}_=, \mathcal{A}_=\}$.

a) A is state-$\mathcal{S}$-equiloaded if and only if a real constant η in $\mathbb{R}$ exists, such that, for all pairs of states p, q in K and for all n in $\mathbb{N}$, the property

$$|\#[p, \mathcal{S}(A, n)] - \#[q, \mathcal{S}(A, n)]| \leq \eta \cdot |\mathcal{S}(A, n)|$$

holds.

b) A is transition-$\mathcal{S}$-equiloaded if and only if a real constant η in $\mathbb{R}$ exists, such that, for all pairs of transitions e, f in D and for all n in $\mathbb{N}$, the property

$$|\#[e, \mathcal{S}(A, n)] - \#[f, \mathcal{S}(A, n)]| \leq \eta \cdot |\mathcal{S}(A, n)|$$

holds.

Proof. We shall prove only the statement for transitions, the proof of the statement for states is analogous.

We shall start with the proof of the easier right-to-left implication. Clearly, since every computation path of length n consists of n transition uses, and since both $\mathcal{C}_=(A,n) = \mathrm{Comp}(A,n)$ and $\mathcal{A}_=(A,n) = \mathrm{Acc}(A,n)$ contain only computation paths of length n, we have

$$\sum_{e\in D} \#[e, \mathcal{S}(A,n)] = n \cdot |\mathcal{S}(A,n)|.$$

Thus,

$$\frac{n}{|D|} \cdot |\mathcal{S}(A,n)| \leq \max_{e\in D} \#[e, \mathcal{S}(A,n)] \leq n \cdot |\mathcal{S}(A,n)|,$$

i.e.,

$$\max_{e\in D} \#[e, \mathcal{S}(A,n)] = g(n)|\mathcal{S}(A,n)|,$$

where $g(n)$ is $\Theta(n)$. Now, if the inequality

$$|\#[e, \mathcal{S}(A,n)] - \#[f, \mathcal{S}(A,n)]| \leq \eta \cdot |\mathcal{S}(A,n)|$$

holds for all e, f in D and for all n in $\mathbb{N}$, then the number of uses of each transition e in D can be written as

$$\#[e, \mathcal{S}(A,n)] = (g(n) + r_e(n))|\mathcal{S}(A,n)|,$$

where $r_e : \mathbb{N} \to \mathbb{R}$ is a function, such that $|r_e(n)| = O(1)$.

Now, if $|\mathcal{S}(A,n)|$ is zero for all $n \geq n_0$ for some n_0 in $\mathbb{N}$, then, applying Lemma 1.5.3,

$$B_A(\mathcal{S}) = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{(g(n) + r_e(n))|\mathcal{S}(A,n)| + 1}{(g(n) + r_f(n))|\mathcal{S}(A,n)| + 1} = \liminf_{n\to\infty} \frac{1}{1} = 1.$$

Thus, in this case the implication holds.

Now, let us suppose that $|\mathcal{S}(A,n_k)| > 0$ for infinitely many nonnegative integers n_k, $k = 0, 1, 2, \ldots$. Then,

$$\begin{aligned} B_A(\mathcal{S}) &= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{(g(n) + r_e(n))|\mathcal{S}(A,n)| + 1}{(g(n) + r_f(n))|\mathcal{S}(A,n)| + 1} = \\ &= \min_{(e,f)\in D^2} \liminf_{k\to\infty} \frac{(g(n_k) + r_e(n_k))|\mathcal{S}(A,n_k)|}{(g(n_k) + r_f(n_k))|\mathcal{S}(A,n_k)|} = \\ &= \min_{(e,f)\in D^2} \liminf_{k\to\infty} \frac{g(n_k) + r_e(n_k)}{g(n_k) + r_f(n_k)} = \min_{(e,f)\in D^2} \liminf_{k\to\infty} \frac{\frac{g(n_k)}{n_k} + \frac{r_e(n_k)}{n_k}}{\frac{g(n_k)}{n_k} + \frac{r_f(n_k)}{n_k}} = 1, \end{aligned}$$

since both $\frac{r_e(n_k)}{n_k}$ and $\frac{r_f(n_k)}{n_k}$ tend to 0 as k goes to infinity and $\frac{g(n_k)}{n_k}$ is bounded from below (and also from above) by a positive constant. Thus, the first implication is proved.

Now, let us prove the remaining left-to-right implication. Let $\mathcal{S}$ be in $\{\mathcal{C}_=, \mathcal{A}_=\}$. Let us suppose that the automaton A is transition-$\mathcal{S}$-equiloaded. We shall show that the alternative definition applies to A.

If the graphical representation of the automaton A does not contain any directed cycle, the implication clearly holds. Let us therefore suppose that there is a directed cycle in the graphical representation of the automaton A.

Let $f : \mathbb{N} \to \mathbb{N}$ denote the function $F_0(n) = |\mathrm{Comp}(A,n)|$ for the case of $\mathcal{S} = \mathcal{C}_=$, and the function $f_0(n) = |\mathrm{Acc}(A,n)|$ for the case of $\mathcal{S} = \mathcal{A}_=$.

Let m be a number of states of the automaton A. For the transition matrix Δ of the automaton A, an $m \times m$ permutation matrix P exists, such that $P \cdot \Delta \cdot P^{-1}$ is in the normal form of a reducible

matrix. That is, for some μ in $\mathbb{N}$,

$$P \cdot \Delta \cdot P^{-1} = \begin{pmatrix} \Delta_{1,1} & \Delta_{1,2} & \dots & \Delta_{1,\mu} \\ \mathbf{0} & \Delta_{2,2} & \dots & \Delta_{2,\mu} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Delta_{\mu,\mu} \end{pmatrix},$$

where $\Delta_{j,j}$ are square blocks for $j = 1, \dots, \mu$ that are either irreducible or 1×1 null matrices. Since the matrices Δ and $P \cdot \Delta \cdot P^{-1}$ are similar, they have the same spectrum. Moreover, since the matrix $P \cdot \Delta \cdot P^{-1}$ is an upper triangular block matrix, its spectrum is the union of the spectra of the blocks $\Delta_{j,j}$, for $j = 1, \dots, \mu$. Thus, also the spectrum of the transition matrix Δ is the union of the spectra of the blocks $\Delta_{j,j}$, for $j = 1, \dots, \mu$.

Let us denote the Perron-Frobenius eigenvalue of the irreducible (or null) block $\Delta_{j,j}$ by ρ'_j (if $\Delta_{j,j}$ is a null block, then $\rho'_j = 0$), for $j = 1, \dots, \mu$. Moreover, let p_j, for $j = 1, \dots, \mu$, denote the greatest natural number, such that

$$\rho'_j, \rho'_j \cdot e^{2\pi i/p_j}, \dots, \rho'_j \cdot e^{2\pi i(p_j-1)/p_j}$$

are eigenvalues of $\Delta_{j,j}$. By the Perron-Frobenius theorem, all these are simple eigenvalues of $\Delta_{j,j}$ (although *not necessarily simple* eigenvalues of Δ). Let us denote the pairwise distinct nonzero eigenvalues ρ, such that $\rho = \rho'_j$ for some $j = 1, \dots, \mu$, by

$$\rho_1, \dots, \rho_r,$$

where r is the number of such pairwise distinct nonzero eigenvalues. Since the graphical representation of the automaton A contains a directed cycle, it follows from Lemma 2.3.4 that $r \geq 1$.

Now, let us consider the spectrum of the transition matrix Δ. Let S be a set of all pairs of nonnegative integers (s_1, s_2) in $\mathbb{N}^2$, such that $s_1 < s_2$, s_1, s_2 are coprime and $\rho_j \cdot e^{2\pi i s_1/s_2}$ is an eigenvalue of Δ, for some j in $\{1, \dots, r\}$. The set S is clearly finite. For given j in $\{1, \dots, r\}$ and (s_1, s_2) in S, let us denote by β_{j,s_1,s_2} the algebraic multiplicity of the eigenvalue $\rho_j \cdot e^{2\pi i s_1/s_2}$ (more precisely, $\rho_j \cdot e^{2\pi i s_1/s_2}$ need not to be an eigenvalue, since the pair (s_1, s_2) in S may not correspond to j – in that case we define $\beta_{j,s_1,s_2} = 0$). Moreover, let us denote the (pairwise distinct) nonzero eigenvalues that are not of this form by $\lambda_1, \dots, \lambda_k$ and their algebraic multiplicities by $\alpha_1, \dots, \alpha_k$, where k in $\mathbb{N}$ is the number of pairwise distinct eigenvalues of Δ that are not of the above defined form. By the results obtained in Section 2.1, for n greater than some n_0 in $\mathbb{N}$ (we shall assume this in the rest of the proof),

$$f(n) = \sum_{j=1}^{r} \sum_{(s_1,s_2)\in S} \sum_{h=0}^{\beta_{j,s_1,s_2}-1} c_{j,s_1,s_2,h} \cdot n^h \cdot \rho_j^n \cdot e^{2\pi i n s_1/s_2} + \sum_{j=1}^{k} \sum_{h=0}^{\alpha_j-1} c_{j,h} \cdot n^h \cdot \lambda_j^n,$$

where

$$c_{j,s_1,s_2,h}, \qquad j = 1, \dots, r, (s_1, s_2) \in S, h = 0, \dots, \beta_{j,s_1,s_2} - 1$$

and

$$c_{j,h}, \qquad j = 1, \dots, k, h = 0, \dots, \alpha_j - 1$$

are constants. Clearly, we may replace each β_{j,s_1,s_2} by the number

$$\beta = \max_{\substack{j=1,\dots,r \\ (s_1,s_2)\in S}} \beta_{j,s_1,s_2}$$

by setting the newly introduced constants $c_{j,s_1,s_2,h}$ to 0. Thus, we obtain

$$f(n) = \sum_{j=1}^{r} \sum_{h=0}^{\beta-1} n^h \cdot \left(\sum_{(s_1,s_2)\in S} c_{j,s_1,s_2,h} \cdot e^{2\pi i n s_1/s_2} \right) \cdot \rho_j^n + \sum_{j=1}^{k} \sum_{h=0}^{\alpha_j-1} c_{j,h} \cdot n^h \cdot \lambda_j^n. \tag{2.8}$$

Next, since $e^{2\pi iN} = 1$ for all nonnegative integers N in $\mathbb{N}$, the function

$$\sum_{(s_1,s_2)\in S} c_{j,s_1,s_2,h} \cdot e^{2\pi i n s_1/s_2} \tag{2.9}$$

(of variable n) is clearly periodic with period s, where s is the least common multiple of all nonnegative integers s_2 in $\mathbb{N}$, such that (s_1, s_2) is in S for some s_1 in $\mathbb{N}$ (the set S is finite, so the least common multiple is well-defined).

Now, for $a = 0, \ldots, s-1$, let us denote by $\varphi_a : \mathbb{N} \to \mathbb{N}$ the function defined by

$$\varphi_a(n) = f(n \cdot s + a)$$

for n in $\mathbb{N}$. From (2.8) and from the periodicity of the function (2.9), it follows that

$$\varphi_a(n) = \sum_{j=1}^{r} \sum_{h=0}^{\beta-1} (n \cdot s + a)^h \cdot C_{j,h} \cdot \rho_j^{ns+a} + \sum_{j=1}^{k} \sum_{h=0}^{\alpha_j - 1} c_{j,h} \cdot (n \cdot s + a)^h \cdot \lambda_j^{ns+a} \tag{2.10}$$

for $a = 0, \ldots, s-1$, where $C_{j,h}$ is a complex constant for $j = 1, \ldots, r$ and $h = 0, \ldots, \beta - 1$. Moreover, in a similar way as in the proof of Lemma 2.3.2, it is possible to prove that the constants $C_{j,h}$ are in fact real.

Let a in $\{0, \ldots, s-1\}$ be fixed. We shall show that if the greatest Perron-Frobenius eigenvalue, for which some nonzero coefficient $C_{j,h}$ exists in (2.10), is ρ_J for some J in $\{1, \ldots, r\}$, then each eigenvalue λ_j (with j in $\{1, \ldots, k\}$) with some nonzero coefficient $c_{j,l}$ (with l in $\{0, \ldots, \alpha_j - 1\}$) in (2.10) is of absolute value strictly smaller then ρ_J.

We shall first prove this for the case $f(n) = F_0(n) = |\mathrm{Comp}(A, n)|$. For the purpose of contradiction, let us suppose that the converse is true, i.e., that an eigenvalue λ_j of absolute value greater than or equal to ρ_J exists, such that $c_{j,l}$ is nonzero in (2.10) for some l. This implies that a computation path γ of the automaton A exists, such that γ visits some state q belonging to the strongly connected component of the graphical representation, to which the eigenvalue λ_j corresponds (if there are more such components, then it visits at least one of them). Otherwise, the function $\varphi_a(n)$ would be the same as the function $\varphi_a(n)$ corresponding to the automaton with this strongly connected component deleted. This argument can be repeated while the reduced automaton has λ_j as an eigenvalue. At the end of this process, we obtain an automaton with the same $\varphi_a(n)$, but without λ_j as an eigenvalue. However, by linear independence of involved functions, this implies that all coefficients $c_{j,l}$, $l = 0, \ldots, \alpha_j - 1$ are zero, i.e., a contradiction.

This strongly connected component has a Perron-Frobenius eigenvalue ρ_b for some b taken from $\{1, \ldots, r\}$. Thus, clearly $\rho_b > |\lambda_j| \geq \rho_J$. Now, if we construct an automaton B from this strongly connected component by choosing q to be its initial state (the set of accepting states may be arbitrary), and if we denote by $\psi(n)$ the number of computation paths of length n in the automaton B, by Lemma 2.3.2 we have $\psi(n) = \Theta(\rho_b^n)$. Thus,

$$\varphi_a(n) = f(n \cdot s + a) = F_0(n \cdot s + a) \geq \psi(n \cdot s + a - |\gamma|) = \Omega\left(\rho_b^{ns+a-|\gamma|}\right),$$

since the computation path γ may continue by all of $\psi(n)$ computation paths of the automaton B. However, this is an obvious contradiction.

We shall omit the proof for the case $f(n) = f_0(n) = |\mathrm{Acc}(A, n)|$. It makes use of Lemma 2.3.1 and Lemma 2.3.3 and may be found in our report [26].

Thus, we may conclude that

$$\varphi_a(n) = \Theta\left((n \cdot s + a)^H \cdot \rho_J^{ns+a}\right), \tag{2.11}$$

where H is a greatest nonnegative integer, such that $c_{J,H}$ is nonzero (by our assumptions, at least one such nonnegative integer exists).

Now, we can finally prove the left-to-right implication from the statement of the theorem. Let $g^{(1)} : \mathbb{N} \to \mathbb{N}$ denote, depending on the case for which the implication is being proved, one of the basic quantities $T_i^e(n) = \#[e, \mathrm{Comp}(A, n)]$, and $t_i^e(n) = \#[e, \mathrm{Acc}(A, n)]$. Let $g^{(2)} : \mathbb{N} \to \mathbb{N}$ denote one of the functions $T_i^f(n) = \#[f, \mathrm{Comp}(A, n)]$, and $t_i^f(n) = \#[f, \mathrm{Acc}(A, n)]$ (depending on the case for which the implication is being proved).

Moreover, let $\varphi_a^{(1)} : \mathbb{N} \to \mathbb{N}$ denote the function

$$\varphi_a^{(1)}(n) = g^{(1)}(n \cdot s + a)$$

for $a = 0, \ldots, s-1$, and $\varphi_a^{(2)} : \mathbb{N} \to \mathbb{N}$ denote the function

$$\varphi_a^{(2)}(n) = g^{(2)}(n \cdot s + a)$$

for $a = 0, \ldots, s-1$. Let a in $\{0, \ldots, s-1\}$ be fixed. By the results obtained in Section 2.1, and by the same argumentation as for the case of functions $f(n)$ and $\varphi_a(n)$, we obtain

$$\varphi_a^{(1)}(n) = \sum_{j=1}^{r} \sum_{h=0}^{2\beta-1} (n \cdot s + a)^h \cdot D_{j,h}^{(1)} \cdot \rho_j^{ns+a} + \sum_{j=1}^{k} \sum_{h=0}^{2\alpha_j-1} d_{j,h}^{(1)} \cdot (n \cdot s + a)^h \cdot \lambda_j^{ns+a}, \tag{2.12}$$

$$\varphi_a^{(2)}(n) = \sum_{j=1}^{r} \sum_{h=0}^{2\beta-1} (n \cdot s + a)^h \cdot D_{j,h}^{(2)} \cdot \rho_j^{ns+a} + \sum_{j=1}^{k} \sum_{h=0}^{2\alpha_j-1} d_{j,h}^{(2)} \cdot (n \cdot s + a)^h \cdot \lambda_j^{ns+a}, \tag{2.13}$$

where $D_{j,h}^{(1)}, D_{j,h}^{(2)}$, $j = 1, \ldots, r$, $h = 0, \ldots, 2\beta - 1$ are real constants, and $d_{j,h}^{(1)}, d_{j,h}^{(2)}$, $j = 1, \ldots, k$, $h = 0, \ldots, 2\alpha_j - 1$ are complex constants.

We assume that the automaton A is transition-$\mathcal{S}$-equiloaded. Moreover, for both $\mathcal{S}$ to which the statement of the theorem applies,

$$\sum_{y \in D} \#[y, \mathcal{S}(A, n)] = n \cdot |\mathcal{S}(A, n)|.$$

Thus, for at least one transition e_g in D, an infinite increasing sequence of nonnegative integers $\{n_k\}_{k=0}^{\infty}$ exists, such that

$$\#[e_g, \mathcal{S}(A, n_k)] \geq \frac{n_k}{|D|} \cdot |\mathcal{S}(A, n_k)|$$

for all k in $\mathbb{N}$. Thus, taking into account (2.11), (2.12), and (2.13), it can be easily seen that if the automaton A is equiloaded, then the constants $D_{J,H+1}^{(1)}$ and $D_{J,H+1}^{(2)}$ have to be both nonzero and equal. Moreover, the constants $D_{J,L}^{(1)}$ and $D_{J,L}^{(2)}$ have to be zero for $L > H + 1$. Thus, the coefficient at $(n \cdot s + a)^h \cdot \rho_J^{ns+a}$ is zero for $h > H$, in $\left|\varphi_a^{(1)}(n) - \varphi_a^{(2)}(n)\right|$. Thus, we have

$$\left|\varphi_a^{(1)}(n) - \varphi_a^{(2)}(n)\right| = O\left((n \cdot s + a)^H \cdot \rho_J^{ns+a}\right).$$

The correctness of the implication then clearly follows from (2.11).

We have proved both implications for both possible choices of $\mathcal{S}$. That is, the theorem is proved. □

It can be easily seen that the equivalence provided by Theorem 2.3.5 does not hold for $\mathcal{S} = \mathcal{C}_{\leq}$ and $\mathcal{S} = \mathcal{A}_{\leq}$ – it is a trivial task to construct a deterministic finite automaton accepting a finite language, such that it satisfies the alternative definition from Theorem 2.3.5, but at the same time, the automaton is not $\mathcal{S}$-equiloaded.

2.3.2 Computation of Basic Quantities via Convolutions

In this subsection, we shall derive an alternative method for computing the basic quantities $\#[e, \mathrm{Comp}(A,n)]$ and $\#[e, \mathrm{Acc}(A,n)]$, for a given transition e, and the quantities $\#[q, \mathrm{Comp}(A,n)]$ and $\#[q, \mathrm{Acc}(A,n)]$, for a given state q. We shall show that these quantities may be computed from certain convolutions. We shall use this alternative method of computation to prove Theorem 2.3.15 on asymptotic properties of the quantities $\#[e, \mathrm{Comp}(A,n)]$ and $\#[e, \mathrm{Acc}(A,n)]$, and, most importantly, to prove the characterization of weakly state-$\mathcal{C}_=$-equiloaded deterministic finite automata (Theorem 2.3.31 and preceeding lemmas).

Notation 2.3.6 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε. In addition to the notation $A_q = (K, \Sigma, \delta, q, F)$, where q is in K, introduced in Section 2.1, we shall use the notation $A^S = (K, \Sigma, \delta, q_0, S)$, where S is a subset of K. That is, by A^S we shall denote the automaton A with the set of accepting states changed to S.

Now, we shall state a lemma that provides an alternative method for computing the number of uses of a given *transition* in computation paths of a given length.

Lemma 2.3.7 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$, for some m in $\mathbb{N}$. Let $e = (q_j, c, q_k)$ in D be a transition, for some c in $\Sigma \cup \{\varepsilon\}$ and j, k in $\{0, 1, \ldots, m-1\}$. Then, for all n in $\mathbb{N}$, the identities

$$\#[e, \mathrm{Comp}(A,n)] = \sum_{i=0}^{n-1} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Comp}(A_{q_k}, n-i-1)|,$$

and

$$\#[e, \mathrm{Acc}(A,n)] = \sum_{i=0}^{n-1} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A_{q_k}, n-i-1)|$$

hold.

Proof. We shall prove only the second identity, since the first identity is clearly its special case: obviously, $\mathrm{Comp}(A,n) = \mathrm{Acc}(A^K, n)$.

The first method how to prove the second identity, is by direct combinatorial insight. In fact, the number of uses of the transition e as an $(i+1)$-th step of accepting computation paths of length n, can be clearly expressed as

$$|\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A_{q_k}, n-i-1)|.$$

The overall number of uses of this transition in accepting computation paths of length n is then clearly the sum of this quantity for all possible i.

However, we shall also present the second proof that does not make use of combinatorial insight. As we have already noted in Section 2.1, the system of OΔEs for

$$t_0^e(n) = \#[e, \mathrm{Acc}(A,n)]$$

can be viewed either as a homogeneous system of $2m$ OΔEs in $2m$ unknown functions, or as a nonhomogeneous system of m OΔEs in m unknown functions. Up to now, we have always used the perspective of the homogeneous system of $2m$ OΔEs. However, if we express the quantity $t_0^e(n)$ by a nonhomogeneous system, we obtain the system

$$\mathbf{t}'_n = \Delta \cdot \mathbf{t}'_{n-1} + \mathbf{x}_{n-1},$$

where

$$\mathbf{t}'_n = \left(t_0^e(n), t_1^e(n), \ldots, t_{m-1}^e(n)\right)^T$$

and

$$\mathbf{x}_n = (\underbrace{0,\ldots,0}_{j-1}, |\mathrm{Acc}(A_{q_k}, n)|, \underbrace{0,\ldots,0}_{m-j})^T. \tag{2.14}$$

The initial conditions are given by

$$\mathbf{t}'_0 = (\underbrace{0,0,\ldots,0}_{m})^T.$$

Thus, it is clear that the column vector $\mathbf{t}'_n$ can be expressed from this nonhomogeneous system of OΔEs as

$$\mathbf{t}'_n = \mathbf{I}_m \cdot \mathbf{x}_{n-1} + \Delta \cdot \mathbf{x}_{n-2} + \ldots + \Delta^{n-2} \cdot \mathbf{x}_1 + \Delta^{n-1} \cdot \mathbf{x}_0 = \sum_{i=0}^{n-1} \Delta^i \cdot \mathbf{x}_{n-i-1}. \tag{2.15}$$

Thus, if we introduce the notation

$$\Delta^i = \begin{pmatrix} d^{(i)}_{0,0} & d^{(i)}_{0,1} & \cdots & d^{(i)}_{0,m-1} \\ d^{(i)}_{1,0} & d^{(i)}_{1,1} & \cdots & d^{(i)}_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ d^{(i)}_{m-1,0} & d^{(i)}_{m-1,1} & \cdots & d^{(i)}_{m-1,m-1} \end{pmatrix}$$

for i in $\mathbb{N}$, from (2.14) and (2.15) we obtain the identity

$$\#[e, \mathrm{Acc}(A, n)] = t^e_0(n) = \sum_{i=0}^{n-1} d^{(i)}_{0,j} \cdot |\mathrm{Acc}(A_{q_k}, n-i-1)|. \tag{2.16}$$

However, if we define

$$\mathbf{y} = (\underbrace{0,\ldots,0}_{j-1}, 1, \underbrace{0,\ldots,0}_{m-j})^T,$$

then $d^{(i)}_{0,j}$ is clearly the first entry of the column vector $\Delta^i \cdot \mathbf{y}$, i.e., by Theorem 2.1.4,

$$d^{(i)}_{0,j} = |\mathrm{Acc}(A^{\{q_j\}}, i)|.$$

Thus, (2.16) can be rewritten as

$$\#[e, \mathrm{Acc}(A, n)] = t^e_0(n) = \sum_{i=0}^{n-1} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A_{q_k}, n-i-1)|.$$

That is, the lemma is proved. □

We shall omit the proof of the next lemma that provides us with an alternative method for computing the quantities $\#[q, \mathrm{Comp}(A, n)]$ and $\#[q, \mathrm{Acc}(A, n)]$, counting the number of uses of a specified *state*. The lemma may be proved analogously as Lemma 2.3.7, or can be proved as a corollary of this lemma. The proof using the second approach can be found in our report [26].

Lemma 2.3.8 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with $K = \{q_0, q_1, \ldots, q_{m-1}\}$, for some m in $\mathbb{N}$. Let q_j in K be a state, for some j in $\{0, 1, \ldots, m-1\}$. Then, for all n in $\mathbb{N}$, the identities

$$\#[q_j, \mathrm{Comp}(A, n)] = \sum_{i=0}^{n} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Comp}(A_{q_j}, n-i)|,$$

and

$$\#[q_j, \mathrm{Acc}(A, n)] = \sum_{i=0}^{n} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A_{q_j}, n-i)|$$

hold.

2.3.3 Asymptotic Properties of Quantities for Strongly Connected Automata

In this subsection, we shall focus on one remarkable property of deterministic finite automata with strongly connected graphical representation. This property is stated in Theorem 2.3.15 and, in the case of automata with strongly connected graphical representation, significantly simplifies the asymptotic estimates for the quantities representing the number of uses of a given transition. This theorem also implies that a similar property holds also for the quantities representing the number of uses of a given state (Corollary 2.3.16).

However, before we proceed to the statement and proof of Theorem 2.3.15, let us briefly introduce the concept of the *period* of a given strongly connected automaton. The period of an automaton is essentialy the same as the period of its transition matrix (see, e.g., [32]). We shall present three alternative characterizations of this concept. We shall omit proofs of Lemma 2.3.9 and of Theorem 2.3.11. The reason for this is that the proofs are lengthy and technical, however they are essentially the same as the proofs of analogous results in the theory of nonnegative matrices (see, e.g., [32]).

Lemma 2.3.9 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation. Then, a positive integer P_A in $\mathbb{N}^+$ exists, such that K can be partitioned into P_A disjoint sets

$$\mathscr{P}(0), \mathscr{P}(1), \ldots, \mathscr{P}(P_A - 1),$$

such that

(i) If, for some q in K, w in Σ^* and n in $\mathbb{N}$, the property

$$(q_0, w) \vdash^n (q, \varepsilon)$$

holds, then q is in $\mathscr{P}(n \bmod P_A)$.

(ii) A nonnegative integer n_0 in $\mathbb{N}$ exists, such that for all $n \geq n_0$ and all q in $\mathscr{P}(n \bmod P_A)$, a word w in Σ^* exists, such that the property

$$(q_0, w) \vdash^n (q, \varepsilon)$$

holds.

Definition 2.3.10 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation. We shall call the positive integer P_A from the previous lemma the *period* of the automaton A.

Theorem 2.3.11 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation. Then, the period P_A of the automaton A is equal to the number of distinct eigenvalues of Δ with the absolute value equal to the spectral radius $\rho(\Delta)$ (i.e., their absolute value is equal to the Perron-Frobenius eigenvalue ρ). This is further equal to the greatest common divisor of lengths of closed directed walks in the graphical representation of the automaton A.

Now we are prepared to proceed to our study of asymptotics for the number of uses of a given transition. However, before we state Theorem 2.3.15, let us prove one lemma that we shall use in its proof.

Lemma 2.3.12 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation. Let $S, T \subseteq \mathscr{P}(k)$ be nonempty sets, for some fixed k in $\{0, 1, \ldots, P_A - 1\}$. Then

$$\left|\mathrm{Acc}(A^S, n)\right| = c \cdot \left|\mathrm{Acc}(A^T, n)\right| \pm o\left(\max\left\{1, \left|\mathrm{Acc}(A^S, n)\right|\right\}\right),$$

for some real constant c in $\mathbb{R}$.

Proof. If the automaton A consists of one isolated state without any transition, the statement of the lemma is trivial. Thus, let us suppose that A has at least one transition. Then, since the graphical representation of the automaton A is strongly connected, the transition matrix Δ is irreducible. Thus, by the Perron-Frobenius Theorem and by the results obtained in Section 2.1, we have

$$\left|\mathrm{Acc}(A^S, n)\right| = \left(\sum_{j=0}^{p-1} c_j \cdot e^{2\pi inj/p}\right) \cdot \rho^n + o(\rho^n)$$

and

$$\left|\mathrm{Acc}(A^T, n)\right| = \left(\sum_{j=0}^{p-1} d_j \cdot e^{2\pi inj/p}\right) \cdot \rho^n + o(\rho^n),$$

where ρ is the Perron-Frobenius eigenvalue of the transition matrix Δ, and where $c_0, \ldots, c_{p-1}$ and $d_0, \ldots, d_{p-1}$ are (in general complex) constants. Further, by Theorem 2.3.11, p is equal to the period P_A of the automaton A.

The functions

$$\sum_{j=0}^{p-1} c_j \cdot e^{2\pi inj/p} \qquad \text{and} \qquad \sum_{j=0}^{p-1} d_j \cdot e^{2\pi inj/p}$$

are clearly periodic with period $p = P_A$. Moreover, in the proof of Lemma 2.3.2, we have observed that these functions are real.

Now, since both S and T are subsets of $\mathscr{P}(k)$, it follows that both $|\mathrm{Acc}(A^S, n)|$ and $|\mathrm{Acc}(A^T, n)|$ are zero for n such that $n \bmod p \neq k$. Moreover, since both S and T are nonempty, it follows from Lemma 2.3.3 that both

$$\sum_{j=0}^{p-1} c_j \cdot e^{2\pi ikj/p} \qquad \text{and} \qquad \sum_{j=0}^{p-1} d_j \cdot e^{2\pi ikj/p}$$

are nonzero. Thus, clearly, the statement of the lemma holds for the real constant c defined by

$$c = \frac{\sum_{j=0}^{p-1} c_j \cdot e^{2\pi ikj/p}}{\sum_{j=0}^{p-1} d_j \cdot e^{2\pi ikj/p}}.$$

Thus, the lemma is proved. □

Before we use this lemma to prove the main result of this subsection, Theorem 2.3.15, let us state one auxiliary lemma and its corollary that we shall use in the proof of Theorem 2.3.15. The proofs of both following statements are easy and left to the reader (however, they may be found in our report [26]).

Lemma 2.3.13 Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of real numbers, such that $a_n \geq 0, b_n > 0$ for all $n \geq n_0$, where n_0 is in $\mathbb{N}$, $b_n \to \infty$ for $n \to \infty$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where L is in $\mathbb{R}$.[7] Then also

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} a_k}{\sum_{k=0}^{n} b_k} = L.$$

Corollary 2.3.14 Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of real numbers, such that $b_n \neq 0$ for all $n \geq n_0$, where n_0 is in $\mathbb{N}$, $b_n \to \infty$ for $n \to \infty$, and

$$a_n = o(b_n).$$

[7] The statement holds also for $L = \pm\infty$, but the case where L is in $\mathbb{R}$ is sufficient for the purposes of this thesis.

Then also

$$\sum_{k=0}^{n} a_k = o\left(\sum_{k=0}^{n} b_k\right).$$

Now we shall proceed to the main result of this subsection. For deterministic finite automata without ε-transitions, and for the case of *accepting* computation paths, we have proved a similar result already in [25]. However, the proof presented in [25] is lengthy, technical, and complicated. The proof presented in what follows is considerably simpler. Thus, the presented theorem may be viewed as a generalization of the result presented in [25], and its proof may be viewed as a significant simplification of the proof from [25].

Theorem 2.3.15 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation. Then for each transition e in D, a real constant a_e in $\mathbb{R}$ exists, such that

$$\#[e, \mathrm{Comp}(A, n)] = (a_e n \pm O(1)) \cdot |\mathrm{Comp}(A, n)|,$$

and

$$\#[e, \mathrm{Acc}(A, n)] = (a_e n \pm O(1)) \cdot |\mathrm{Acc}(A, n)|.$$

Proof. To prove the theorem, it clearly suffices to show that a real constant a_e in $\mathbb{R}$ exists, such that

$$\#[e, \mathrm{Acc}(A^S, n)] = (a_e n \pm O(1)) \cdot |\mathrm{Acc}(A^S, n)| \tag{2.17}$$

for all sets of states $S \subseteq K$. If S is empty, then both $|\mathrm{Acc}(A^S, n)|$ and $\#[e, \mathrm{Acc}(A^S, n)]$ are zero and the property is trivially satisfied for all constants a_e. Thus, we shall assume that S is nonempty.

We shall suppose that A has at least one transition – otherwise, the claim is trivial. Then, since the graphical representation of the automaton A is strongly connected, the transition matrix Δ is irreducible. Thus, it follows from the results obtained in Section 2.1, from Theorem 2.3.11, and from Lemma 2.3.3, that

$$\#[e, \mathrm{Acc}(A^S, n)] = \left(\sum_{j=0}^{P_A - 1} c_j \cdot e^{2\pi i n j / P_A}\right) \cdot n \cdot \rho^n \pm O(1) \cdot |\mathrm{Acc}(A^S, n)|,$$

where ρ is the Perron-Frobenius eigenvalue of the transition matrix Δ, and where $c_0, \ldots, c_{P_A - 1}$ in $\mathbb{C}$ are constants. Moreover, as we have observed in the proof of Lemma 2.3.2, the periodic function

$$\sum_{j=0}^{P_A - 1} c_j \cdot e^{2\pi i n j / P_A}$$

is always real, and, by Lemma 2.3.3, $|\mathrm{Acc}(A^S, n_k)| = \Theta(\rho^{n_k})$, where $\{n_k\}_{k=0}^{\infty}$ is the infinite increasing sequence of all nonnegative integers n, such that $|\mathrm{Acc}(A^S, n)|$ is nonzero (such a sequence exists, since the graphical representation of A is strongly connected, at least one transition exists, and the set S is nonempty). Thus, we may restate this as follows: a sequence $\{b_n^S\}_{n=0}^{\infty}$ periodic with period P_A exists, such that

$$\#[e, \mathrm{Acc}(A^S, n)] = (b_n^S n \pm O(1)) \cdot |\mathrm{Acc}(A^S, n)|. \tag{2.18}$$

We shall first prove that a real constant a_e in $\mathbb{R}$ exists, such that (2.17) holds for all sets S, such that $S \subseteq \mathscr{P}(k)$ for some fixed k in $\{0, 1, \ldots, P_A - 1\}$. In that case, $|\mathrm{Acc}(A^S, n)|$ is zero for all n in $\mathbb{N}$, such that $n \bmod P_A \neq k$. Thus, clearly, for every such given S, the property

$$\#[e, \mathrm{Acc}(A^S, n)] = (b_k^S n \pm O(1)) \cdot |\mathrm{Acc}(A^S, n)|$$

holds. We shall prove that b_k^S is the same for all such S, and that will be our constant a_e.

First, we shall prove that $b_0^S = b_0^T$ for all nonempty $S, T \subseteq \mathscr{P}(0)$. By Lemma 2.3.12,

$$|\mathrm{Acc}(A^S, n)| = c \cdot \left|\mathrm{Acc}(A^T, n)\right| \pm o\left(\max\left\{1, \left|\mathrm{Acc}(A^S, n)\right|\right\}\right). \tag{2.19}$$

By Lemma 2.3.7, and subsequently by (2.19), Corollary 2.3.14, and once again Lemma 2.3.7,

$$\begin{aligned}
\#[e, \mathrm{Acc}(A^S, n)] &= \sum_{i=0}^{n-1} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A^S_{q_k}, n-i-1)| = \\
&= \sum_{i=0}^{n-1} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot \Big(c \cdot |\mathrm{Acc}(A^T_{q_k}, n-i-1)| \pm \\
&\qquad \pm o\left(\max\left\{1, |\mathrm{Acc}(A^S_{q_k}, n-i-1)|\right\}\right)\Big) = \\
&= \left(\sum_{i=0}^{n-1} c \cdot |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A^T_{q_k}, n-i-1)|\right) \pm O(1) \cdot |\mathrm{Acc}(A^S, n)| = \\
&= \left(c \cdot \sum_{i=0}^{n-1} |\mathrm{Acc}(A^{\{q_j\}}, i)| \cdot |\mathrm{Acc}(A^T_{q_k}, n-i-1)|\right) \pm O(1) \cdot |\mathrm{Acc}(A^S, n)| = \\
&= c \cdot \#[e, \mathrm{Acc}(A^T, n)] \pm O(1) \cdot |\mathrm{Acc}(A^S, n)|.
\end{aligned}$$

This clearly implies that $b_0^S = b_0^T =: b_0$.

Now, let $T \subseteq \mathscr{P}(k)$ for some k in $\{1, 2, \ldots, P_A - 1\}$. We shall show that $b_k^T = b_0$. By what we have proved up to now, it clearly suffices to show that $b_k^{\mathscr{P}(k)} = b_0$. However,

$$\begin{aligned}
\#[e, \mathrm{Acc}(A^{\mathscr{P}(k)}, n)] &= \sum_{q \in \mathscr{P}(0)} \Big(\#[e, \mathrm{Acc}(A^{\{q\}}, n-k)] \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)| + \\
&\qquad + |\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot \#[e, \mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)]\Big) = \\
&= \sum_{q \in \mathscr{P}(0)} \Big(\#[e, \mathrm{Acc}(A^{\{q\}}, n-k)] \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|\Big) + \\
&+ \sum_{q \in \mathscr{P}(0)} \Big(|\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot \#[e, \mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)]\Big).
\end{aligned} \tag{2.20}$$

Moreover, clearly,

$$|\mathrm{Acc}(A^{\mathscr{P}(k)}, n)| = \sum_{q \in \mathscr{P}(0)} |\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|.$$

Thus, by what we have proved above,

$$\begin{aligned}
&\sum_{q \in \mathscr{P}(0)} \Big(\#[e, \mathrm{Acc}(A^{\{q\}}, n-k)] \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|\Big) = \\
&\quad = \sum_{q \in \mathscr{P}(0)} \Big((b_0(n-k) \pm O(1)) \cdot |\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|\Big) = \\
&\quad = \sum_{q \in \mathscr{P}(0)} \Big((b_0 n \pm O(1)) \cdot |\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|\Big) = \\
&\quad = (b_0 n \pm O(1)) \cdot \sum_{q \in \mathscr{P}(0)} |\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)| = \\
&\quad = (b_0 n \pm O(1)) \cdot |\mathrm{Acc}(A^{\mathscr{P}(k)}, n)|.
\end{aligned} \tag{2.21}$$

Moreover, since $\#[e, \mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)]$ is a constant that is zero whenever $|\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|$ is zero, we have

$$\begin{aligned}\sum_{q\in\mathscr{P}(0)} \Big(|\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot \#[e, \mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)]\Big) = \\ = O\left(\sum_{q\in\mathscr{P}(0)} |\mathrm{Acc}(A^{\{q\}}, n-k)| \cdot |\mathrm{Acc}(A_q^{\mathscr{P}(k)}, k)|\right) = \\ = O(|\mathrm{Acc}(A^{\mathscr{P}(k)}, n)|) = O(1) \cdot |\mathrm{Acc}(A^{\mathscr{P}(k)}, n)|. \qquad (2.22)\end{aligned}$$

Now, by plugging (2.21) and (2.22) into (2.20), we obtain

$$\#[e, \mathrm{Acc}(A^{\mathscr{P}(k)}, n)] = (b_0 n \pm O(1)) \cdot |\mathrm{Acc}(A^{\mathscr{P}(k)}, n)|.$$

Thus, we have proved that for all sets of states T, such that $T \subseteq \mathscr{P}(k)$ for some k in the set $\{0, 1, \ldots, P_A - 1\}$, the property

$$\#[e, \mathrm{Acc}(A^T, n)] = (b_0 n \pm O(1)) \cdot |\mathrm{Acc}(A^T, n)|.$$

holds. It remains to prove that the property holds also for all other sets of states S. However, this is clear, since for n such that $n \bmod P_A = k$,

$$\begin{aligned}\#[e, \mathrm{Acc}(A^S, n)] = \#[e, \mathrm{Acc}(A^{S\cap\mathscr{P}(k)}, n)] = (b_0 n \pm O(1)) \cdot |\mathrm{Acc}(A^{S\cap\mathscr{P}(k)}, n)| = \\ = (b_0 n \pm O(1)) \cdot |\mathrm{Acc}(A^S, n)|.\end{aligned}$$

Since this holds for all k in $\{0, 1, \ldots, P_A - 1\}$, the theorem is proved. □

Theorem 2.3.15 directly implies the following corollary for the quantities representing the numbers of uses of states.

Corollary 2.3.16 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with strongly connected graphical representation. Then for each state q in K, a real constant a_q in $\mathbb{R}$ exists, such that

$$\#[q, \mathrm{Comp}(A, n)] = (a_q n \pm O(1)) \cdot |\mathrm{Comp}(A, n)|,$$

and

$$\#[q, \mathrm{Acc}(A, n)] = (a_q n \pm O(1)) \cdot |\mathrm{Acc}(A, n)|.$$

Proof. In Theorem 2.1.9, we have observed that the identities

$$\begin{aligned}\#[q, \mathrm{Comp}(A, n)] = \sum_{(p,c,q)\in D} \#[(p, c, q), \mathrm{Comp}(A, n)], \\ \#[q, \mathrm{Acc}(A, n)] = \sum_{(p,c,q)\in D} \#[(p, c, q), \mathrm{Acc}(A, n)]\end{aligned}$$

hold for all states $q \neq q_0$, and that for the state q_0, the identities

$$\begin{aligned}\#[q_0, \mathrm{Comp}(A, n)] = |\mathrm{Comp}(A, n)| + \sum_{(p,c,q_0)\in D} \#[(p, c, q_0), \mathrm{Comp}(A, n)], \\ \#[q_0, \mathrm{Acc}(A, n)] = |\mathrm{Acc}(A, n)| + \sum_{(p,c,q_0)\in D} \#[(p, c, q_0), \mathrm{Acc}(A, n)]\end{aligned}$$

hold (the sums go through all p in K and c in $\Sigma \cup \{\varepsilon\}$). Thus, as a direct consequence, the claim holds for

$$a_q = \sum_{(p,c,q)\in D} a_{(p,c,q)}.$$

That is, the corollary is proved. □

2.3.4 Almost-Equivalence of Transition-Equiloadedness $\mathcal{S}$-Measures

In this subsection, we shall use Theorem 2.3.15 to prove Theorem 2.3.17 that will be of key importance for the theory of transition-equiloaded DFA and DFAε. We shall prove that the transition-equiloadedness $\mathcal{S}$-measures are *almost* equivalent for $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$. These measures are not equivalent only for DFAε that do not contain any reachable directed cycle from which at least one accepting state is reachable, i.e., for a subset of DFAε accepting finite languages.

This fact can be viewed as a justification of the definition of transition-equiloadedness, since it shows a certain robustness of this definition – no matter which of the standard parameters $\mathcal{S}$ is chosen, there are only minor differences in the resulting family of automata and languages.

However, for the case of state-equiloadedness, this property does not hold. We shall prove that the values of state-equiloadedness $\mathcal{S}$-measures may significantly vary for different $\mathcal{S}$ also for automata, for which the transition-equiloadedness measures have to be the same.

Theorem 2.3.17 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε, such that in the graphical representation of A, there is at least one reachable directed cycle from which an accepting state is reachable. Then,

$$B_A(\mathcal{C}_=) = B_A(\mathcal{A}_=) = B_A(\mathcal{C}_\leq) = B_A(\mathcal{A}_\leq).$$

Proof. First, let us suppose that the graphical representation of the automaton A is *not* strongly connected. Then, there is at least one transition e in D, such that

$$\#[e, \gamma] \leq 1$$

for all computation paths γ. Thus,

$$\#[e, \mathrm{Comp}(A, n)] = O(1) \cdot |\mathrm{Comp}(A, n)| \tag{2.23}$$

and

$$\#[e, \mathrm{Acc}(A, n)] = O(1) \cdot |\mathrm{Acc}(A, n)|. \tag{2.24}$$

On the other hand, clearly,

$$\sum_{y \in D} \#[y, \mathrm{Comp}(A, n)] = n \cdot |\mathrm{Comp}(A, n)|$$

and

$$\sum_{y \in D} \#[y, \mathrm{Acc}(A, n)] = n \cdot |\mathrm{Acc}(A, n)|.$$

Thus, transitions f_1, f_2 in D and infinite increasing sequences $\{n_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$ of nonnegative integers have to exist (since there is a reachable directed cycle in the graphical representation of A, from which an accepting state is reachable), such that

$$\begin{aligned} |\mathrm{Comp}(A, n_k)| &> 0, \\ |\mathrm{Acc}(A, m_k)| &> 0, \\ \#[f_1, \mathrm{Comp}(A, n_k)] &\geq \frac{n_k}{|D|} \cdot |\mathrm{Comp}(A, n_k)|, \end{aligned}$$

and

$$\#[f_2, \mathrm{Acc}(A, m_k)] \geq \frac{m_k}{|D|} \cdot |\mathrm{Acc}(A, m_k)|$$

for all k in $\mathbb{N}$. Thus, by Lemma 1.5.3,

$$B_A(\mathcal{C}_=) \leq \liminf_{n \to \infty} \frac{\#[e, \mathrm{Comp}(A, n)] + 1}{\#[f_1, \mathrm{Comp}(A, n)] + 1} \leq \liminf_{k \to \infty} \frac{\#[e, \mathrm{Comp}(A, n_k)] + 1}{\#[f_1, \mathrm{Comp}(A, n_k)] + 1} = 0,$$

and

$$B_A(\mathcal{A}_=) \leq \liminf_{n\to\infty} \frac{\#[e, \mathrm{Acc}(A,n)]+1}{\#[f_2, \mathrm{Acc}(A,n)]+1} \leq \liminf_{k\to\infty} \frac{\#[e, \mathrm{Acc}(A,m_k)]+1}{\#[f_2, \mathrm{Acc}(A,m_k)]+1} = 0.$$

Thus, $B_A(\mathcal{C}_=) = B_A(\mathcal{A}_=) = 0$. Moreover, clearly,

$$\#[e, \mathrm{Comp}(A,n_k)] = o\left(\#[f_1, \mathrm{Comp}(A,n_k)]\right),$$

and

$$\#[e, \mathrm{Acc}(A,m_k)] = o\left(\#[f_2, \mathrm{Acc}(A,m_k)]\right).$$

Thus, by Corollary 2.3.14,

$$\sum_{j=0}^{k} \#[e, \mathrm{Comp}(A,n_j)] = o\left(\sum_{j=0}^{k} \#[f_1, \mathrm{Comp}(A,n_j)]\right), \tag{2.25}$$

and

$$\sum_{j=0}^{k} \#[e, \mathrm{Acc}(A,m_j)] = o\left(\sum_{j=0}^{k} \#[f_2, \mathrm{Acc}(A,m_j)]\right). \tag{2.26}$$

Now, let us assume that the sequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ contain *all* nonnegative integers, such that the conditions imposed on them are satisfied. Then, by what we have observed in the proof of Theorem 2.3.5, for some k_0 in $\mathbb{N}$, the sequences $\{n_k\}_{k=k_0}^{\infty}$ and $\{m_k\}_{k=k_0}^{\infty}$ are periodic with period s in $\mathbb{N}$. Thus, adding terms $\#[e, \mathrm{Comp}(A,l)]$ resp. $\#[e, \mathrm{Acc}(A,l)]$ for l not in $\{n_k\}_{k=0}^{\infty}$ resp. $\{m_k\}_{k=0}^{\infty}$ to the sum on the left side of (2.25) resp. (2.26) will preserve the asymptotic relation. Thus, we obtain

$$\#[e, \mathrm{Comp}(A, \leq n_k)] = o\left(\sum_{j=0}^{k} \#[f_1, \mathrm{Comp}(A,n_j)]\right),$$

and

$$\#[e, \mathrm{Acc}(A, \leq m_k)] = o\left(\sum_{j=0}^{k} \#[f_2, \mathrm{Acc}(A,m_j)]\right),$$

and, as a direct consequence,

$$\#[e, \mathrm{Comp}(A, \leq n)] + 1 = o\left(\#[f_1, \mathrm{Comp}(A, \leq n)] + 1\right),$$

and

$$\#[e, \mathrm{Acc}(A, \leq n)] + 1 = o\left(\#[f_2, \mathrm{Acc}(A, \leq n)] + 1\right).$$

But this is equivalent to that $B_A(\mathcal{C}_\leq) = B_A(\mathcal{A}_\leq) = 0$. Thus, we have proved that if the graphical representation of the automaton A is not strongly connected, then the property

$$B_A(\mathcal{C}_=) = B_A(\mathcal{A}_=) = B_A(\mathcal{C}_\leq) = B_A(\mathcal{A}_\leq) = 0$$

holds.

Now, let us suppose that the graphical representation of the automaton A *is* strongly connected. Then, by Theorem 2.3.15, for each transition e in D a constant a_e in $\mathbb{R}$ exists, such that

$$\#[e, \mathrm{Comp}(A,n)] = (a_e n \pm O(1)) \cdot |\mathrm{Comp}(A,n)|,$$

and

$$\#[e, \mathrm{Acc}(A,n)] = (a_e n \pm O(1)) \cdot |\mathrm{Acc}(A,n)|.$$

Let us denote

$$B = \min_{(e,f)\in D^2} \frac{a_e}{a_f}.$$

By Lemma 1.5.3,

$$B_A(\mathcal{C}_=) = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\#[e, \mathrm{Comp}(A,n)]+1}{\#[f, \mathrm{Comp}(A,n)]+1} =$$
$$= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{(a_e n \pm O(1))|\mathrm{Comp}(A,n)|+1}{(a_f n \pm O(1))|\mathrm{Comp}(A,n)|+1} = \min_{(e,f)\in D^2} \frac{a_e}{a_f} = B,$$

and

$$B_A(\mathcal{A}_=) = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\#[e, \mathrm{Acc}(A,n)]+1}{\#[f, \mathrm{Acc}(A,n)]+1} =$$
$$= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{(a_e n \pm O(1))|\mathrm{Acc}(A,n)|+1}{(a_f n \pm O(1))|\mathrm{Acc}(A,n)|+1} = \min_{(e,f)\in D^2} \frac{a_e}{a_f} = B.$$

Moreover, by Lemma 1.5.3,

$$B_A(\mathcal{C}_\leq) = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\#[e, \mathrm{Comp}(A,\leq n)]+1}{\#[f, \mathrm{Comp}(A,\leq n)]+1} = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n} \#[e, \mathrm{Comp}(A,k)]+1}{\sum_{k=0}^{n} \#[f, \mathrm{Comp}(A,k)]+1} =$$
$$= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n} ((a_e k \pm O(1)) \cdot |\mathrm{Comp}(A,k)|)+1}{\sum_{k=0}^{n} ((a_f k \pm O(1)) \cdot |\mathrm{Comp}(A,k)|)+1} =$$
$$= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{a_e \cdot \sum_{k=0}^{n} ((k \pm O(1)) \cdot |\mathrm{Comp}(A,k)|)+1}{a_f \cdot \sum_{k=0}^{n} ((k \pm O(1)) \cdot |\mathrm{Comp}(A,k)|)+1} = \min_{(e,f)\in D^2} \frac{a_e}{a_f} = B.$$

Similarly,

$$B_A(\mathcal{A}_\leq) = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\#[e, \mathrm{Acc}(A,\leq n)]+1}{\#[f, \mathrm{Acc}(A,\leq n)]+1} = \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n} \#[e, \mathrm{Acc}(A,k)]+1}{\sum_{k=0}^{n} \#[f, \mathrm{Acc}(A,k)]+1} =$$
$$= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n} ((a_e k \pm O(1)) \cdot |\mathrm{Acc}(A,k)|)+1}{\sum_{k=0}^{n} ((a_f k \pm O(1)) \cdot |\mathrm{Acc}(A,k)|)+1} =$$
$$= \min_{(e,f)\in D^2} \liminf_{n\to\infty} \frac{a_e \cdot \sum_{k=0}^{n} ((k \pm O(1)) \cdot |\mathrm{Acc}(A,k)|)+1}{a_f \cdot \sum_{k=0}^{n} ((k \pm O(1)) \cdot |\mathrm{Acc}(A,k)|)+1} = \min_{(e,f)\in D^2} \frac{a_e}{a_f} = B.$$

Thus, we have proved that if the graphical representation of the automaton A is strongly connected, then

$$B_A(\mathcal{C}_=) = B_A(\mathcal{A}_=) = B_A(\mathcal{C}_\leq) = B_A(\mathcal{A}_\leq) = B.$$

That is, the theorem is proved. □

As we have already anticipated, the situation is far more complicated for the case of state-$\mathcal{S}$-equiloadedness. In the following example, we shall show an example of a deterministic finite automaton, for which the properties proved for transition-equiloadedness measures are violated for state-equiloadedness measures. To be more specific, we shall construct a deterministic finite automaton A, such that state-equiloadedness measures $\beta_A(\mathcal{C}_=), \beta_A(\mathcal{A}_=), \beta_A(\mathcal{C}_\leq)$, and $\beta_A(\mathcal{A}_\leq)$ attain pairwise distinct values.

Example 2.3.18 Let us consider a deterministic finite automaton $A = (K, \Sigma, \delta, q_0, F)$ defined as follows: $K = \{q_0, q_1, q_2, q_3, q_4, q_5\}$, $\Sigma = \{a, b, c, d\}$, $F = \{q_2, q_3, q_5\}$, and

$$\begin{array}{lllll}
\delta(q_0,a) = q_1, & \delta(q_0,b) = q_1, & \delta(q_0,c) = q_2, & \delta(q_0,d) = q_4, & \delta(q_1,a) = q_0, \\
\delta(q_1,b) = q_0, & \delta(q_2,a) = q_3, & \delta(q_2,b) = q_3, & \delta(q_3,a) = q_2, & \delta(q_3,b) = q_2, \\
\delta(q_4,a) = q_5, & \delta(q_5,a) = q_4, & \delta(q_5,b) = q_4, & \delta(q_5,c) = q_4, & \delta(q_5,d) = q_4.
\end{array}$$

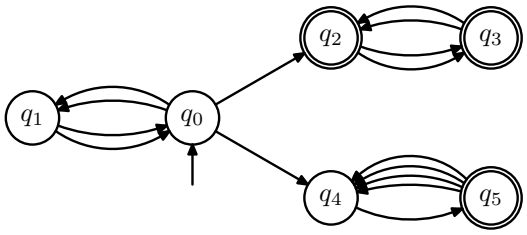


Figure 2.2: The automaton A. Since the number of transitions is relatively high, and since the characters are not important in this example, character labels are omitted in the diagram.

The graphical representation of the automaton A is depicted in Figure 2.2 (without labels at transitions specifying characters – this example works no matter what characters are used in the transitions, and there is not enough space in the figure to include these labels).

By implementing the method for computing the closed form of the basic quantities, presented in Section 2.1, on a computer, we have computed the state-equiloadedness $\mathcal{S}$-measures of the automaton A (the details of this computation may be found in our report [26]). The results are as follows:

$$\beta_A(\mathcal{C}_=) = \frac{1}{3}, \qquad \beta_A(\mathcal{A}_=) = 0, \qquad \beta_A(\mathcal{C}_\leq) = \frac{2}{5}, \qquad \beta_A(\mathcal{A}_\leq) = \frac{1}{7}.$$

That is, these four state-equiloadedness $\mathcal{S}$-measures are pairwise different.

2.3.5 Characterizations of Weak $\mathcal{S}$-Equiloadedness for several $\mathcal{S}$

In this subsection, we shall prove characterizations of weakly transition-$\mathcal{S}$-equiloaded and weakly state-$\mathcal{S}$-equiloaded DFA and DFAε, for several possible choices of $\mathcal{S}$.

We shall start with the characterization of weak transition-equiloadedness for $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=\}$. In [25], we have presented the characterization of weak transition-$\mathcal{A}_=$-equiloadedness for DFA (without ε-transitions). The proof presented in this thesis is based on the proof from [25], however is slightly different. The remaining characterizations presented in this subsection are entirely new.

Theorem 2.3.19 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with connected graphical representation.

a) A is weakly transition-$\mathcal{C}_=$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle, or it is strongly connected.

b) A is weakly transition-$\mathcal{A}_=$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle from which some accepting state is reachable, or it is strongly connected.

Proof. First, we shall prove the easier left-to-right implications (i.e., the *only if* part). Let the automaton A be weakly transition-$\mathcal{A}_=$-equiloaded. If the graphical representation of the automaton A does not contain any reachable directed cycle from which some accepting state is reachable, the number of accepting computation paths is finite and the implication holds trivially.

Now, let the graphical representation of the automaton A contain such directed cycle, i.e., the number of accepting computation paths be infinite. For the purpose of contradiction, let us suppose that the graphical representation of the automaton A is not strongly connected. Since the graphical representation is connected, this implies that a bridge has to exist in the graphical representation.

In other words, a transition e in D exists, such that

$$\#[e, \mathrm{Acc}(A, n)] \leq |\mathrm{Acc}(A, n)|$$

for all n in $\mathbb{N}$. However, since the number of accepting computation paths is infinite and since

$$\sum_{y \in D} \#[y, \mathrm{Acc}(A, n)] = n \cdot |\mathrm{Acc}(A, n)|,$$

a transition f in D and an infinite increasing sequence of nonnegative integers $\{n_k\}_{k=0}^{\infty}$ have to exist, such that $|\mathrm{Acc}(A, n_k)| > 0$ and

$$\#[f, \mathrm{Acc}(A, n_k)] \geq \frac{n_k}{|D|} \cdot |\mathrm{Acc}(A, n_k)|$$

for all k in $\mathbb{N}$. However, existence of such a pair of transitions clearly implies that the transition-equiloadedness $\mathcal{A}_=$-measure of the automaton A is zero.

The case $\mathcal{S} = \mathcal{C}_=$ is analogous.

Now, let us prove the more difficult right-to-left implications. First, let the graphical representation of the automaton A be without a reachable directed cycle from which some accepting state is reachable. Then, there is only a finite number of accepting computation paths. As a consequence, a nonnegative integer n_0 in $\mathbb{N}$ exists, such that there is no accepting computation path of length n, for all nonnegative integers $n \geq n_0$. Thus, both the numerator and the denominator of the n-th transition-equiloadedness $\mathcal{A}_=$-quotient are 1, for all $n \geq n_0$. This implies that $B_A(\mathcal{A}_=) = 1 > 0$, i.e., the automaton A is transition-$\mathcal{A}_=$-equiloaded, and thus also weakly transition-$\mathcal{A}_=$-equiloaded.

Further, let us suppose that the graphical representation of the automaton A does not contain any reachable directed cycle. Then a nonnegative integer n_0 in $\mathbb{N}$ exists, such that there is not any (accepting or nonaccepting) computation path of length n, for all nonnegative integers $n \geq n_0$. Thus, again, both the numerator and the denominator of the n-th transition-equiloadedness $\mathcal{C}_=$-quotient are 1, for all $n \geq n_0$. Thus, the automaton A is transition-$\mathcal{C}_=$-equiloaded, and therefore also weakly transition-$\mathcal{C}_=$-equiloaded.

Now, we shall prove that if the graphical representation of the automaton A is strongly connected, then the automaton A is both $\mathcal{C}_=$-equiloaded and $\mathcal{A}_=$-equiloaded.

Let us start with the proof of $\mathcal{C}_=$-equiloadedness. First, we shall prove that a constant M in $\mathbb{R}$ exists, such that for all nonnegative integers $n \geq |K|$, all transitions e, f in D and for all states q in K, the property

$$\#_{A_q}[f, \mathrm{Comp}(A_q, n)] \leq M \cdot \#_{A_q}[e, \mathrm{Comp}(A_q, n)] \tag{2.27}$$

holds (as above in this thesis, A_q denotes the automaton A with its initial state changed to q). To be more specific, we shall define the constant M as follows. Since the graphical representation of the automaton A is strongly connected, also the graphical representation of A_p is strongly connected, for each p in K. Thus, by Lemma 2.3.2,

$$|\mathrm{Comp}(A_p, n)| = \Theta(\rho^n)$$

for each p in K. Therefore, it follows that a real number $M' \geq 1$ exists, such that

$$\max_{p \in K} |\mathrm{Comp}(A_p, n)| \leq M' \cdot \min_{p \in K} |\mathrm{Comp}(A_p, n)|$$

for all n in $\mathbb{N}$. We define M by

$$M = 2|K||\Sigma|^{2|K|} \cdot M'.$$

We shall prove the inequality (2.27) by a variant of the mathematical induction. In essence, it shall be an induction on n, but in one induction step, we shall prove the property for $|K|$ values of n at once. Let the transitions e, f in D be fixed.

1. Let n be in $\{|K|, |K|+1, \ldots, 2|K|-1\}$. Since the graphical representation of the automaton A is strongly connected, it is clearly possible to reach the initial state of the transition e in at most $|K|-1$ steps from the state q. One step is needed to pass this transition and the

corresponding computation path can be arbitrarily prolonged – thus, it can be prolonged also to the length of n.

That is, we have proved that

$$\#_{A_q}[e, \mathrm{Comp}(A_q, n)] \geq 1.$$

However, on the other hand, clearly

$$\#_{A_q}[f, \mathrm{Comp}(A_q, n)] \leq 2|K||\Sigma|^{2|K|}$$

(since in each state there are at most $|\Sigma|$ transitions through which the computation path can proceed – this holds also for DFAε, since the ε-transition can lead from the state only if there is no other transition leading from that state). Thus,

$$\#_{A_q}[f, \mathrm{Comp}(A_q, n)] \leq 2|K||\Sigma|^{2|K|} \cdot \#_{A_q}[e, \mathrm{Comp}(A_q, n)].$$

That is, since $2|K||\Sigma|^{2|K|} \leq M$, the basis of the induction is proved. However, we shall keep in mind that for n in $\{|K|, |K|+1, \ldots, 2|K|-1\}$, we can use also $2|K||\Sigma|^{2|K|}$ instead of M – we shall use this fact in the proof of the induction step.

2. Let us suppose that (2.27) holds for all n in $\{|K|, |K|+1, \ldots, k|K|-1\}$. We shall prove that the property (2.27) holds also for all n in the set $\{k|K|, k|K|+1, \ldots, (k+1)|K|-1\}$.

   Let n be in $\{k|K|, k|K|+1, \ldots, (k+1)|K|-1\}$. Every computation path γ' of the automaton A_q of length n can be decomposed into a computation path γ of length $|K|$ ending in some state $p(\gamma)$ and a computation path of the automaton $A_{p(\gamma)}$ of length $n-|K|$. Thus, if we denote by $p(\gamma)$ the state, in which the computation path γ ends, we have

$$\begin{aligned}
\#_{A_q}[f, \mathrm{Comp}(A_q, n)] = & \sum_{\gamma \in \mathrm{Comp}(A_q, |K|)} \Big(\#_{A_q}[f, \gamma] \cdot |\mathrm{Comp}(A_{p(\gamma)}, n-|K|)| + \\
& + \#_{A_{p(\gamma)}}[f, \mathrm{Comp}(A_{p(\gamma)}, n-|K|)] \Big) \leq \\
\leq & \sum_{\gamma \in \mathrm{Comp}(A_q, |K|)} \Big(\#_{A_q}[f, \gamma] \cdot \max_{p \in K} |\mathrm{Comp}(A_p, n-|K|)| + \\
& + \#_{A_{p(\gamma)}}[f, \mathrm{Comp}(A_{p(\gamma)}, n-|K|)] \Big) \overset{IH}{\leq} \\
\overset{IH}{\leq} & \#_{A_q}[f, \mathrm{Comp}(A_q, |K|)] \cdot \max_{p \in K} |\mathrm{Comp}(A_p, n-|K|)| + \\
& + \sum_{\gamma \in \mathrm{Comp}(A_q, |K|)} M \cdot \#_{A_{p(\gamma)}}[e, \mathrm{Comp}(A_{p(\gamma)}, n-|K|)],
\end{aligned}$$

   where IH stands for the application of the induction hypothesis. Thus, by applying the result obtained in the basis of the induction to the quantity $\#_{A_q}[f, \mathrm{Comp}(A_q, |K|)]$, we may continue the derivation as

$$\begin{aligned}
\#_{A_q}[f, \mathrm{Comp}(A_q, n)] \leq & 2|K||\Sigma|^{2|K|} \cdot \#_{A_q}[e, \mathrm{Comp}(A_q, |K|)] \cdot \max_{p \in K} |\mathrm{Comp}(A_p, n-|K|)| + \\
& + M \cdot \sum_{\gamma \in \mathrm{Comp}(A_q, |K|)} \#_{A_{p(\gamma)}}[e, \mathrm{Comp}(A_{p(\gamma)}, n-|K|)]
\end{aligned}$$

   and since, as we have already noted,

$$\max_{p \in K} |\mathrm{Comp}(A_p, n-|K|)| \leq \frac{M}{2|K||\Sigma|^{2|K|}} \cdot \min_{p \in K} |\mathrm{Comp}(A_p, n-|K|)|,$$

we obtain

$$\begin{aligned}
\#_{A_q}[f,\mathrm{Comp}(A_q,n)] &\leq M\cdot\#_{A_q}[e,\mathrm{Comp}(A_q,|K|)]\cdot\min_{p\in K}|\mathrm{Comp}(A_p,n-|K|)|+\\
&\quad+M\cdot\sum_{\gamma\in\mathrm{Comp}(A_q,|K|)}\#_{A_{p(\gamma)}}[e,\mathrm{Comp}(A_{p(\gamma)},n-|K|)]=\\
&=M\cdot\sum_{\gamma\in\mathrm{Comp}(A_q,|K|)}\Big(\#_{A_q}[e,\gamma]\cdot\min_{p\in K}|\mathrm{Comp}(A_p,n-|K|)|+\\
&\quad+\#_{A_{p(\gamma)}}[e,\mathrm{Comp}(A_{p(\gamma)},n-|K|)]\Big)\leq\\
&\leq M\cdot\sum_{\gamma\in\mathrm{Comp}(A_q,|K|)}\Big(\#_{A_q}[e,\gamma]\cdot|\mathrm{Comp}(A_{p(\gamma)},n-|K|)|+\\
&\quad+\#_{A_{p(\gamma)}}[e,\mathrm{Comp}(A_{p(\gamma)},n-|K|)]\Big)=\\
&=M\cdot\#_{A_q}[e,\mathrm{Comp}(A_q,n)].
\end{aligned}$$

Thus, we have proved that the inequality (2.27) holds for all nonnegative integers $n \geq |K|$, all transitions e, f in D and for all states q in K. However, by Lemma 1.5.3,

$$\begin{aligned}
B_A(\mathcal{C}_=) &= \min_{(e,f)\in D^2}\liminf_{n\to\infty}\frac{\#[e,\mathrm{Comp}(A,n)]+1}{\#[f,\mathrm{Comp}(A,n)]+1}\geq\\
&\geq\min_{(e,f)\in D^2}\liminf_{n\to\infty}\frac{\#[e,\mathrm{Comp}(A,n)]+1}{M\cdot\#[e,\mathrm{Comp}(A,n)]+1}=\frac{1}{M}>0,
\end{aligned}$$

since $\#[e,\mathrm{Comp}(A,n)] \to \infty$ for $n \to \infty$ and for all e in D. Thus, the automaton A is weakly transition-$\mathcal{C}_=$-equiloaded.

It remains to prove the implication for $\mathcal{S} = \mathcal{A}_=$. Let e, f in D be transitions. Let $\{n_k\}_{k=0}^{\infty}$ be the infinite increasing sequence of all nonnegative integers n, such that $|\mathrm{Acc}(A,n)| > 0$. First, we shall show that a constant s in $\mathbb{N}$ exists, such that for all nonnegative integers k, $n_k \geq s$, the property

$$\#[f,\mathrm{Comp}(A,n_k-s)]\leq\#[f,\mathrm{Acc}(A,n_k)] \tag{2.28}$$

holds. To prove this, let S be the set of all lengths of closed walks in the graphical representation of the automaton A, beginning and ending in the vertex corresponding to the initial state q_0. Let d be the greatest common divisor of elements of S. Clearly, the set S is closed under addition and thus, by Lemma 2.3.1, contains all multiples of d greater than some nonnegative integer N in $\mathbb{N}$.

Moreover, let us define the set R as follows: a nonnegative integer n in $\mathbb{N}$ is in R, if r in $\{0,1,\ldots,d-1\}$ exists, such that n is the smallest nonnegative integer with residue r after the division by d, such that $(q_0,w)\vdash^n(q,\varepsilon)$ for some word w in Σ^* and accepting state q in F. It is clear that R is a finite set, since it contains at most d elements. Moreover, it is clear that if n_k has a residue r after the division by d for some k in $\mathbb{N}$, then there exists an element $R(n_k)$ in R with the residue r as well. Let us denote the maximal element of R by R_{max}.

We shall prove that the inequality (2.28) holds for

$$s = N + |K| + R_{max}.$$

It clearly suffices to show that every computation path γ of length $n_k - s$ can be prolonged to at least one accepting computation path of length n_k. In fact:

1. The initial state q_0 can be reached from the resulting state of the computation path γ in at most $|K|-1$ steps. If we denote the corresponding prolonged computation path by γ', then $|\gamma'|$ has to be divisible by d, since the walk in the graphical representation of A corresponding to the computation path γ' is obviously closed.

2. Since $n_k - R(n_k)$ and thus also $(n_k - R(n_k)) - |\gamma'|$ are divisible by d, and since the inequality $(n_k - R(n_k)) - |\gamma'| \geq N$ holds, the computation path γ' can be prolonged to a computation path γ'' of length $n_k - R(n_k)$ ending in q_0.
3. From the definition of the set R, it follows that the computation path γ'' can be prolonged to a computation path of length n_k, ending in an accepting state.

Thus, we have proved that (2.28) holds. Moreover, clearly

$$\#[f, \mathrm{Acc}(A, n_k)] \leq \#[f, \mathrm{Comp}(A, n_k)].$$

for all k in $\mathbb{N}$. Thus, for all k in $\mathbb{N}$, such that $n_k \geq s$, we have

$$\#[f, \mathrm{Comp}(A, n_k - s)] \leq \#[f, \mathrm{Acc}(A, n_k)] \leq \#[f, \mathrm{Comp}(A, n_k)]. \tag{2.29}$$

Once again, for a given computation path γ, let us denote by $p(\gamma)$ the resulting state of this computation. Now, for k in $\mathbb{N}$, such that $n_k \geq s$, we have

$$\begin{aligned}
\#[f, \mathrm{Comp}(A, n_k)] = &\sum_{\gamma \in \mathrm{Comp}(A, n_k - s)} \left(\#[f, \gamma] \cdot |\mathrm{Comp}(A_{p(\gamma)}, s)| + \#_{A_{p(\gamma)}}[f, \mathrm{Comp}(A_{p(\gamma)}, s)] \right) \leq \\
&\leq \#[f, \mathrm{Comp}(A, n_k - s)] \cdot \max_{p \in K} |\mathrm{Comp}(A_p, s)| + \\
&\quad + \sum_{\gamma \in \mathrm{Comp}(A, n_k - s)} \#_{A_{p(\gamma)}}[f, \mathrm{Comp}(A_{p(\gamma)}, s)] \leq \\
&\leq \#[f, \mathrm{Comp}(A, n_k - s)] \cdot \max_{p \in K} |\mathrm{Comp}(A_p, s)| + \\
&\quad + |\mathrm{Comp}(A, n_k - s)| \cdot \max_{p \in K} \#_{A_p}[f, \mathrm{Comp}(A_p, s)].
\end{aligned}$$

Now, both

$$\max_{p \in K} |\mathrm{Comp}(A_p, s)| \qquad \text{and} \qquad \max_{p \in K} \#_{A_p}[f, \mathrm{Comp}(A_p, s)]$$

are constants (they do not depend on n). Thus, if we denote by Q the greater of these two constants, we have

$$\#[f, \mathrm{Comp}(A, n_k)] \leq Q \cdot (\#[f, \mathrm{Comp}(A, n_k - s)] + |\mathrm{Comp}(A, n_k - s)|).$$

However, since we have already proved that the automaton with strongly connected graphical representation is weakly transition-$\mathcal{C}_=$-equiloaded, clearly

$$\#[f, \mathrm{Comp}(A, n_k - s)] = \omega(|\mathrm{Comp}(A, n_k - s)|),$$

and thus,

$$\#[f, \mathrm{Comp}(A, n_k)] \leq 2Q \cdot \#[f, \mathrm{Comp}(A, n_k - s)] \tag{2.30}$$

for n_k greater than some N' in $\mathbb{N}$. Thus, for n_k greater than some N_0 in $\mathbb{N}$, we have

$$\begin{aligned}
\#[f, \mathrm{Acc}(A, n_k)] &\overset{(2.29)}{\leq} \#[f, \mathrm{Comp}(A, n_k)] \overset{(2.30)}{\leq} 2Q \cdot \#[f, \mathrm{Comp}(A, n_k - s)] \overset{(2.27)}{\leq} \\
&\overset{(2.27)}{\leq} 2MQ \cdot \#[e, \mathrm{Comp}(A, n_k - s)] \overset{(2.29)}{\leq} 2MQ \cdot \#[e, \mathrm{Acc}(A, n_k)],
\end{aligned}$$

since (2.29) holds also for $f = e$. Thus, by Lemma 1.5.3,

$$\begin{aligned}
B_A(\mathcal{A}_=) &= \min_{(e,f) \in D^2} \liminf_{n \to \infty} \frac{\#[e, \mathrm{Acc}(A, n)] + 1}{\#[f, \mathrm{Acc}(A, n)] + 1} \geq \\
&\geq \min_{(e,f) \in D^2} \liminf_{n \to \infty} \frac{\#[e, \mathrm{Acc}(A, n)] + 1}{2MQ \cdot \#[e, \mathrm{Acc}(A, n)] + 1} = \frac{1}{2MQ} > 0,
\end{aligned}$$

i.e., the automaton A is weakly transition-$\mathcal{A}_{=}$-equiloaded. The theorem is proved. □

In the theorem that follows, we shall prove the characterization of weakly transition-$\mathcal{C}_{\leq}$-equiloaded and weakly transition-$\mathcal{A}_{\leq}$-equiloaded deterministic finite automata. We shall prove that for DFA and DFAε, the weak transition-$\mathcal{C}_{\leq}$-equiloadedness is equivalent to the weak transition-$\mathcal{C}_{=}$-equiloadedness, and that the weak transition-$\mathcal{A}_{\leq}$-equiloadedness is equivalent to the weak transition-$\mathcal{A}_{=}$-equiloadedness.

Theorem 2.3.20 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε with connected graphical representation.

a) A is weakly transition-$\mathcal{C}_{\leq}$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle, or is strongly connected. That is, the automaton A is weakly transition-$\mathcal{C}_{\leq}$-equiloaded, if and only if the automaton A is weakly transition-$\mathcal{C}_{=}$-equiloaded.

b) A is weakly transition-$\mathcal{A}_{\leq}$-equiloaded, if and only if its graphical representation either does not contain any reachable directed cycle from which some accepting state is reachable, or is strongly connected. That is, the automaton A is weakly transition-$\mathcal{A}_{\leq}$-equiloaded, if and only if the automaton A is weakly transition-$\mathcal{A}_{=}$-equiloaded.

Proof. First, let us suppose that the graphical representation of the automaton A does not contain any reachable directed cycle from which some accepting state is reachable. Then, there is only a finite number of accepting computation paths of the automaton A, and thus, a nonnegative integer n_0 in $\mathbb{N}$ exists, such that for all $n \geq n_0$ and for each transition e in D, the property

$$\#[e, \mathrm{Acc}(A, \leq n)] = \#[e, \mathrm{Acc}(A, \leq n_0)]$$

holds. Thus,

$$B_A(\mathcal{A}_{\leq}) = \min_{(e,f)\in D^2} \frac{\#[e, \mathrm{Acc}(A, \leq n_0)] + 1}{\#[f, \mathrm{Acc}(A, \leq n_0)] + 1} > 0,$$

i.e., the automaton A is weakly transition-$\mathcal{A}_{\leq}$-equiloaded.

Similarly, let us suppose that the graphical representation of the automaton A does not contain any reachable directed cycle. Then, by the same reasoning as above,

$$B_A(\mathcal{C}_{\leq}) = \min_{(e,f)\in D^2} \frac{\#[e, \mathrm{Comp}(A, \leq n_0)] + 1}{\#[f, \mathrm{Comp}(A, \leq n_0)] + 1} > 0,$$

and the automaton A is weakly transition-$\mathcal{C}_{\leq}$-equiloaded.

Now, let us suppose that the automaton A contains at least one reachable directed cycle from which some accepting state is reachable. Then, by Theorem 2.3.17, the equality

$$B_A(\mathcal{A}_{\leq}) = B_A(\mathcal{A}_{=})$$

holds. Thus, by Theorem 2.3.19, it follows that the automaton A is weakly transition-$\mathcal{A}_{\leq}$-equiloaded, if and only if its graphical representation is strongly connected.

Finally, let us suppose that the automaton A contains at least one reachable directed cycle. Then, for each transition e in D and all n in $\mathbb{N}$, the property

$$\#[e, \mathrm{Comp}(A, \leq n)] = \#[e, \mathrm{Acc}(A^K, \leq n)]$$

holds, where $A^K = (K, \Sigma, \delta, q_0, K)$ is the automaton A with all states accepting. Thus, by Theorem 2.3.17, we obtain

$$B_A(\mathcal{C}_{\leq}) = B_{A^K}(\mathcal{A}_{\leq}) = B_{A^K}(\mathcal{A}_{=}).$$

Since the graphical representation of the automaton A is strongly connected if and only if the graphical representation of the automaton A^K is strongly connected, it follows from Theorem

2.3.19 that the automaton A is weakly transition-$\mathcal{C}_{\leq}$-equiloaded, if and only if its graphical representation is strongly connected. Thus, the theorem is proved. □

Before we turn our attention to weakly state-$\mathcal{S}$-equiloaded DFA and DFAε, we shall state a theorem that we have proved in [25] for weakly transition-$\mathcal{A}_{=}$-equiloaded DFA (without ε-transitions), and that can be clearly extended to hold also for another choices of $\mathcal{S}$. Since the original proof is not very complicated, we shall not present it here. We shall prove only the extension of the theorem to $\mathcal{S}$ in $\{\mathcal{C}_{=}, \mathcal{C}_{\leq}, \mathcal{A}_{\leq}\}$.

Theorem 2.3.21 Let L in $\mathscr{R}$ be a regular language. Let $\mathcal{S}$ be in $\{\mathcal{C}_{=}, \mathcal{A}_{=}, \mathcal{C}_{\leq}, \mathcal{A}_{\leq}\}$. The language L is a weakly transition-$\mathcal{S}$-equiloaded DFA-language, if and only if the minimal DFA accepting L is weakly transition-$\mathcal{S}$-equiloaded.

Proof. The proof for $\mathcal{S} = \mathcal{A}_{=}$ can be found in [25]. By Theorem 2.3.20, the theorem holds also for $\mathcal{S} = \mathcal{A}_{\leq}$.

Now, by Theorem 2.3.19, it is clear that $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_{=}) \subseteq \mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{=})$ (in fact, we shall prove in Subsection 2.3.6 that these families are equal). That is, if L is in $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_{=})$, then it is also in $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{=})$ and thus, the minimal DFA accepting L is weakly transition-$\mathcal{A}_{=}$-equiloaded. Moreover, it is clear that if the graphical representation of the minimal DFA accepting L does not contain any reachable directed cycle from which some accepting state is reachable, then it also does not contain any other reachable directed cycle – otherwise, it would be possible to delete the states of the cycle without changing the accepted language, and that would be a contradiction with the assumption that the given automaton is minimal. However, this property together with Theorem 2.3.19 implies that this minimal DFA is also weakly transition-$\mathcal{C}_{=}$-equiloaded. Since the converse implication is trivial, the theorem holds also for the case $\mathcal{S} = \mathcal{C}_{=}$.

The last remaining case, $\mathcal{S} = \mathcal{C}_{\leq}$, follows directly from the case $\mathcal{S} = \mathcal{C}_{=}$ and from Theorem 2.3.20. □

In what follows, we shall characterize the family of weakly state-$\mathcal{C}_{=}$-equiloaded deterministic finite automata. However, before we state this characterization (Theorem 2.3.31), we shall introduce some notation and prove several technical lemmas that we shall use in its proof.

Notation 2.3.22 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε, such that its graphical representation has k strongly connected components, corresponding to disjoint sets of states $K_0, K_1, \ldots, K_{k-1}$, such that

$$\bigcup_{i=0}^{k-1} K_i = K,$$

and q_0 is in K_0. Let i be in $\{0, 1, \ldots, k-1\}$, and q be in K_i. Then, by $A[K_i, q]$, we denote the automaton obtained by the restriction of the set of states of A to K_i, and by choosing q as an initial state of the resulting automaton. Formally, $A[K_i, q] = (K_i, \Sigma, \delta', q, F \cap K_i)$, where, for p, q in K_i and c in Σ, $\delta'(p, c) = q$ whenever $\delta(p, c) = q$.

Definition 2.3.23 Let the symbols have the same meaning as in Notation 2.3.22. We shall define the *SCC-dag* $SCC(A)$ of the automaton A as follows: $SCC(A)$ is a directed acyclic graph with the set of vertices $\{K_0, K_1, \ldots, K_{k-1}\}$, and with an edge from the vertex K_i to the vertex K_j (for some i, j in $\{0, 1, \ldots, k-1\}$), if and only if states p in K_i and q in K_j exist, such that $(p, c) \vdash (q, \varepsilon)$ for some c in $\Sigma \cup \{\varepsilon\}$.

Notation 2.3.24 Let the symbols have the same meaning as above. Let $x = (K_{i_1}, K_{i_2}, \ldots, K_{i_s})$ be a path in $SCC(A)$. Let q be a state in K_{i_1}. Then, by $\mathrm{Comp}_x(A_q, n)$, we denote the set of computation paths γ of the automaton A_q, such that γ follows the path x, i.e., it first visits some nonzero number of states in K_{i_1}, then some nonzero number of states in K_{i_2}, etc., and finally ends in some state in K_{i_s}. Similarly, by $\mathrm{Acc}_x(A_q, n)$, we denote the set of all such accepting computation paths.

Notation 2.3.25 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε, let K_i and K_j be vertices of its SCC-dag. Then, by $\mathrm{Pth}[K_i, K_j]$, we denote the set of all paths in $SCC(A)$, from the vertex K_i to the vertex K_j.

Lemma 2.3.26 Let the symbols have the same meaning as above. Let $x = (K_{i_1}, K_{i_2}, \ldots, K_{i_s})$ be a path in $SCC(A)$. Let q be a state in K_{i_1}. For $j = 0, 1, \ldots, k-1$, let us denote by ρ_j the Perron-Frobenius eigenvalue of the transition matrix $\Delta_{A[K_j, q_j]}$ of the automaton $A[K_j, q_j]$, where q_j is an arbitrary[8] state in K_j, for $j = 1, \ldots, k-1$. Let us denote by ρ_{max} the greatest of the Perron-Frobenius eigenvalues corresponding to strongly connected components on the path x, i.e.,

$$\rho_{max} = \max_{j=1,\ldots,s} \rho_{i_j}.$$

Then, if $\rho_{max} > 0$, the number of computation paths in $\mathrm{Comp}_x(A_q, n)$ is asymptotically equal to

$$|\mathrm{Comp}_x(A_q, n)| = \Theta\left(n^{r-1} \cdot \rho_{max}^n\right),$$

where r is the number of strongly connected components corresponding to vertices on x, such that their Perron-Frobenius eigenvalue is ρ_{max}, i.e.,

$$r = \left| \left\{ j \in \{1, \ldots, s\} \;\middle|\; \rho_{i_j} = \rho_{max} \right\} \right|.$$

If $\rho_{max} = 0$, then a nonnegative integer constant R exists, such that

$$|\mathrm{Comp}_x(A_q, n)| = R \cdot [n = s].$$

Proof. For $j = 1, \ldots, s$, let us denote by x_j the path $x_j = (K_{i_1}, K_{i_2}, \ldots, K_{i_j})$. We shall prove by induction on j that the statement of the lemma holds for all x_j, $j = 1, \ldots, s$.

For this purpose, let us denote by $\rho_{max}^{(j)}$ the greatest of the Perron-Frobenius eigenvalues corresponding to strongly connected components on the path x_j, and by $r^{(j)}$, the number of strongly connected components corresponding to vertices on x_j, such that their Perron-Frobenius eigenvalue is $\rho_{max}^{(j)}$. That is, $\rho_{max}^{(j)}$ and $r^{(j)}$ are defined analogously as ρ_{max} and r, but for the path x_j instead of x.

1. For $j = 1$, it is obvious that the property

   $$|\mathrm{Comp}_{x_1}(A_q, n)| = |\mathrm{Comp}(A[K_{i_1}, q], n)|$$

   holds. However, the graphical representation of the automaton $A[K_{i_1}, q]$ is strongly connected, and thus, by Lemma 2.3.2,

   $$|\mathrm{Comp}(A[K_{i_1}, q], n)| = \Theta(\rho_{i_1}^n).$$

   However, since K_{i_1} is the only vertex on the path x_1, clearly $\rho_{max}^{(1)} = \rho_{i_1}$, and $r^{(1)} = 1$. Thus, the basis of the induction is proved.

2. Let us suppose that the statement of the lemma holds for all x_j, $j = 1, \ldots, t$, for some t in $\{1, \ldots, s-1\}$. We shall prove that it holds also for x_{t+1}. Let q_{trans} in K_{i_t} be a state, such that a state q'_{trans} in $K_{i_{t+1}}$ exists, such that

   $$(q_{trans}, c) \vdash (q'_{trans}, \varepsilon)$$

   for some c in $\Sigma \cup \{\varepsilon\}$. Let $P = P_{A[K_{i_t}, q_t]}$ be the period of the automaton $A[K_{i_t}, q_t]$. Then, obviously, at least one number p in $\{0, 1, \ldots, P-1\}$ and a nonnegative integer n_0 in $\mathbb{N}$ exists, such that

   $$\mathrm{Acc}_{x_t}(A_q^{\{q_{trans}\}}, n \cdot P + p)$$

[8] This eigenvalue is obviously the same, no matter what state q_j in K_j is chosen.

is nonempty for all $n \geq n_0$. Thus, it can be easily seen (a similar reasoning has been used a number of times up to now in this thesis) that a constant C in $\mathbb{N}$ exists, such that for $n \geq n_0$, every computation path of length $(n - C) \cdot P + p$ can be prolonged to a computation path of length $n \cdot P + p$, ending in q_{trans}. In other words,

$$\left|\mathrm{Acc}_{x_t}(A_q^{\{q_{trans}\}}, n \cdot P + p)\right| \geq \left|\mathrm{Comp}_{x_t}(A_q, (n - C) \cdot P + p)\right| \geq \\ \geq C' \cdot \left|\mathrm{Comp}_{x_t}(A_q, n \cdot P + p)\right|, \tag{2.31}$$

for some positive real constant C' in $\mathbb{R}^+$ (the second inequality is a direct consequence of the induction hypothesis).

Now, it is obvious that the number of computation paths in $\mathrm{Comp}_{x_{t+1}}(A_q, n)$ is greater than or equal to the number of uses of the transition $(q_{trans}, c, q'_{trans})$ in these computation paths. Moreover, by a direct combinatorial insight, it is clear that Lemma 2.3.7 can be generalized to hold also for $\mathrm{Comp}_{x_{t+1}}(A_q, n)$ instead of $\mathrm{Comp}(A, n)$. That is,

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| \geq \#\left[(q_{trans}, c, q'_{trans}), \mathrm{Comp}_{x_{t+1}}(A_q, n)\right] = \\ = \sum_{i=0}^{n-1} \left|\mathrm{Acc}_{x_t}(A_q^{\{q_{trans}\}}, i)\right| \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - i - 1)\right|. \tag{2.32}$$

Further, from the induction hypothesis and Lemma 2.3.2, it is clear that a positive real constant D in $\mathbb{R}^+$ exists, such that

$$\left|\mathrm{Comp}_{x_t}(A_q, i \cdot P + p)\right| \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - (i \cdot P + p) - 1)\right| \geq \\ \geq D \cdot \sum_{l=0}^{2P-1} \left|\mathrm{Comp}_{x_t}(A_q, i \cdot P + l)\right| \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - (i \cdot P + l) - 1)\right|. \tag{2.33}$$

Thus, with use of inequalities (2.31) and (2.33), we may continue[9] the derivation (2.32) as follows:

$$\begin{aligned}
\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| &\geq \sum_{i=0}^{n-1} \left|\mathrm{Acc}_{x_t}(A_q^{\{q_{trans}\}}, i)\right| \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - i - 1)\right| \geq \\
&\geq \sum_{i=0}^{\lfloor (n-1-p)/P \rfloor} \left|\mathrm{Acc}_{x_t}(A_q^{\{q_{trans}\}}, i \cdot P + p)\right| \cdot \\
&\qquad \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - (i \cdot P + p) - 1)\right| \overset{(2.31)}{\geq} \\
&\overset{(2.31)}{\geq} C' \cdot \sum_{i=0}^{\lfloor (n-1-p)/P \rfloor} \left|\mathrm{Comp}_{x_t}(A_q^{\{q_{trans}\}}, i \cdot P + p)\right| \cdot \\
&\qquad \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - (i \cdot P + p) - 1)\right| \overset{(2.33)}{\geq} \\
&\overset{(2.33)}{\geq} C' \cdot D \cdot \sum_{i=0}^{n-1} \left|\mathrm{Comp}_{x_t}(A_q, i)\right| \cdot \left|\mathrm{Comp}(A[K_{i_{t+1}}, q'_{trans}], n - i - 1)\right|.
\end{aligned}$$

Now, let us first examine the case $0 < \rho_{i_{t+1}} = \rho_{max}^{(t)} = \rho_{max}^{(t+1)} =: \rho$. Then, $r^{(t+1)} = r^{(t)} + 1$,

[9] At least for automata with infinite number of accepting computation paths in $\mathrm{Acc}_{x_t}(A_q^{\{q_{trans}\}})$. However, the case of automata with finite number of such computation paths is trivial.

and by the use of induction hypothesis and Lemma 2.3.2, we obtain

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| \geq C' \cdot D \cdot \sum_{i=0}^{n-1} \Theta\left(i^{r^{(t)}-1} \cdot \rho^i\right) \cdot \Theta\left(\rho^{n-i-1}\right) =$$
$$= E \cdot \rho^n \sum_{i=0}^{n-1} i^{r^{(t)}-1} = \Theta\left(n^{r^{(t+1)}-1} \cdot \rho^n\right).$$

where E in $\mathbb{R}^+$ is a positive real constant. Thus, in this case,

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| = \Omega\left(n^{r^{(t+1)}-1} \cdot \rho^n\right).$$

We consider to be obvious that the theorem holds also for the case $0 = \rho_{i_{t+1}} = \rho_{max}^{(t)} = \rho_{max}^{(t+1)}$.

Now, let us examine the case $\rho_{i_{t+1}} = \rho_{max}^{(t+1)} > \rho_{max}^{(t)} > 0$. Then, $r^{(t+1)} = 1$. The reader may easily convince himself that for arbitrary two distinct real numbers, a, b in $\mathbb{R}$, $a > b$, the formula

$$\sum_{i=0}^{n} a^i b^{n-i-1} = \frac{-a}{b-a} \cdot a^n + \frac{b}{b-a} \cdot b^n = \Theta(a^n) \tag{2.34}$$

holds. We obtain

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| \geq C' \cdot D \cdot \sum_{i=0}^{n-1} \Theta\left(i^{r^{(t)}-1} \cdot \left(\rho_{max}^{(t)}\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right).$$

For some $\varepsilon > 0$, such that $\rho_{max}^{(t)} + \varepsilon < \rho_{i_{t+1}}$, we have

$$\sum_{i=0}^{n-1} \Theta\left(\left(\rho_{max}^{(t)}\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right) \preceq$$
$$\preceq \sum_{i=0}^{n-1} \Theta\left(i^{r^{(t)}-1} \cdot \left(\rho_{max}^{(t)}\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right) \preceq$$
$$\preceq \sum_{i=0}^{n-1} \Theta\left(\left(\rho_{max}^{(t)} + \varepsilon\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right).$$

However, both

$$\sum_{i=0}^{n-1} \Theta\left(\left(\rho_{max}^{(t)}\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right),$$

and

$$\sum_{i=0}^{n-1} \Theta\left(\left(\rho_{max}^{(t)} + \varepsilon\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right)$$

are $\Theta\left(\rho_{i_{t+1}}^{n}\right)$, by (2.34). Thus, also

$$\sum_{i=0}^{n-1} \Theta\left(i^{r^{(t)}-1} \cdot \left(\rho_{max}^{(t)}\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right) = \Theta\left(\rho_{i_{t+1}}^{n}\right),$$

and that implies

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| = \Omega\left(n^{r^{(t+1)}-1} \cdot \left(\rho_{max}^{(t+1)}\right)^n\right).$$

The proof of the previous case with $\rho_{max}^{(t)} = 0$ is left to the reader.

Finally, let us examine the case $0 < \rho_{i_{t+1}} < \rho_{max}^{(t+1)} = \rho_{max}^{(t)}$. In this case, we have $r^{(t+1)} = r^{(t)}$. By a slightly more involved process than in the previous case, it is possible to prove that

$$\sum_{i=0}^{n-1} \Theta\left(i^{r^{(t)}-1} \cdot \left(\rho_{max}^{(t)}\right)^i\right) \cdot \Theta\left(\rho_{i_{t+1}}^{n-i-1}\right) = \Theta\left(n^{r^{(t)}-1} \cdot \left(\rho_{max}^{(t)}\right)^n\right),$$

and thus also

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| = \Omega\left(n^{r^{(t+1)}-1} \cdot \left(\rho_{max}^{(t+1)}\right)^n\right).$$

The proof of the previous case with $\rho_{i_{t+1}} = 0$ is left to the reader, once again.

Thus, we have proved that

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| = \Omega\left(n^{r^{(t+1)}-1} \cdot \left(\rho_{max}^{(t+1)}\right)^n\right)$$

holds for all three possible cases, and it remains to prove that

$$\left|\mathrm{Comp}_{x_{t+1}}(A_q, n)\right| = O\left(n^{r^{(t+1)}-1} \cdot \left(\rho_{max}^{(t+1)}\right)^n\right).$$

However, this asymptotic relation may be proved in a similar manner (this estimate is easier) and the details are left to the reader.

The lemma is proved. □

Lemma 2.3.27 Let the symbols have the same meaning as in Notation 2.3.22. Let q be a state in K_j, for some j in $\{0, 1, \ldots, k-1\}$. Let s be in $\{0, 1, \ldots, k-1\}$. Then,

$$\left|\mathrm{Acc}(A_q^{K_s}, n)\right| = \sum_{x \in \mathrm{Pth}[K_j, K_s]} \left|\mathrm{Comp}_x(A_q, n)\right| = \Theta\left(\max_{x \in \mathrm{Pth}[K_j, K_s]} \left|\mathrm{Comp}_x(A_q, n)\right|\right).$$

Proof. It is clear that every computation path γ in $\mathrm{Acc}(A_q^{K_s}, n)$ is also in $\mathrm{Comp}_x(A_q, n)$, for some x in $\mathrm{Pth}[K_j, K_s]$. The converse is also obviously true. Moreover, if γ_1 is in $\mathrm{Comp}_x(A_q, n)$, and γ_2 is in $\mathrm{Comp}_y(A_q, n)$, for some x, y in $\mathrm{Pth}[K_j, K_s]$, $x \neq y$, then clearly $\gamma_1 \neq \gamma_2$. That is, the first identity is proved.

Now, obviously,

$$\max_{x \in \mathrm{Pth}[K_j, K_s]} \left|\mathrm{Comp}_x(A_q, n)\right| \geq \frac{1}{|\mathrm{Pth}[K_j, K_s]|} \cdot \sum_{x \in \mathrm{Pth}[K_j, K_s]} \left|\mathrm{Comp}_x(A_q, n)\right|,$$

and

$$\max_{x \in \mathrm{Pth}[K_j, K_s]} \left|\mathrm{Comp}_x(A_q, n)\right| \leq \sum_{x \in \mathrm{Pth}[K_j, K_s]} \left|\mathrm{Comp}_x(A_q, n)\right|,$$

and since $1/|\mathrm{Pth}[K_j, K_s]|$ is a constant, the second asymptotic identity is proved as well. Thus, the lemma is proved. □

Notation 2.3.28 Let the symbols have the same meaning as in Notation 2.3.22. Let K_i, K_j, K_l be vertices of $SCC(A)$. Then, by $\mathrm{Pth}[K_i, K_j, K_l]$, we shall denote the set of all paths $x = (K_{i_1}, K_{i_2}, \ldots, K_{i_s})$ in $SCC(A)$, such that r in $\{1, 2, \ldots, s\}$ exists, such that $(K_{i_1}, \ldots, K_{i_r})$ is in $\mathrm{Pth}[K_i, K_j]$, and $(K_{i_r}, \ldots, K_{i_s})$ is in $\mathrm{Pth}[K_j, K_l]$. That is, by $\mathrm{Pth}[K_i, K_j, K_l]$ we denote the set of all paths x in $SCC(A)$ from the vertex K_i to the vertex K_l, such that the vertex K_j lies on x.

Furthermore, we shall use a notation

$$\mathrm{Pth}[K_i, K_j, *] = \bigcup_{l=0}^{k-1} \mathrm{Pth}[K_i, K_j, K_l].$$

Lemma 2.3.29 Let the symbols have the same meaning as in Notation 2.3.22. Let q be a state in K_j, for some j in $\{0,1,\ldots,k-1\}$. Let s be a member of the set $\{0,1,\ldots,k-1\}$. Let us denote by ρ_j the Perron-Frobenius eigenvalue corresponding to K_j, and by ρ_{max} the greatest of the Perron-Frobenius eigenvalues corresponding to strongly connected components on paths in $\mathrm{Pth}[K_0, K_j, K_s]$. Let us denote by r the greatest positive integer, such that a path x in $\mathrm{Pth}[K_0, K_j, K_s]$ exists, such that there are r strongly connected components on x with the Perron-Frobenius eigenvalue equal to ρ_{max}. Then:

(i) If $\rho_{max} > 0$, and $\rho_j = \rho_{max}$, then

$$\#\left[q, \mathrm{Acc}(A^{K_s}, n)\right] = \Theta\left(n^r \cdot \rho_{max}^n\right).$$

(ii) If $\rho_{max} > 0$, and $\rho_j < \rho_{max}$, then

$$\#\left[q, \mathrm{Acc}(A^{K_s}, n)\right] = \Theta\left(n^{r-1} \cdot \rho_{max}^n\right).$$

(iii) If $\rho_{max} = \rho_j = 0$, then

$$\#\left[q, \mathrm{Acc}(A^{K_s}, n)\right] = 0$$

for all n greater than some n_0 in $\mathbb{N}$.

Proof. We shall prove these three claims separately:

(i) In this case, there has to be a path x_1 in $\mathrm{Pth}[K_0, K_j]$ with r_1 strongly connected components with Perron-Frobenius eigenvalue ρ_{max}, and a path x_2 in $\mathrm{Pth}[K_j, K_s]$ with r_2 strongly connected components with Perron-Frobenius eigenvalue ρ_{max}, so that $r_1 + r_2 = r + 1$. Moreover, since K_j lies both on x_1 and x_2, it follows that $r_1 > 0$, and also $r_2 > 0$.

Thus, from Lemma 2.3.26 and Lemma 2.3.27, it follows that

$$\left|\mathrm{Acc}(A^{K_j}, n)\right| = \Theta\left(n^{r_1-1} \cdot \rho_{max}^n\right).$$

Similarly,

$$\left|\mathrm{Acc}(A_q^{K_s}, n)\right| = \Theta\left(n^{r_2-1} \cdot \rho_{max}^n\right).$$

Moreover, by a standard reasoning concerning the period of the j-th strongly connected component, it is possible to prove that

$$\sum_{i=0}^{n} \left|\mathrm{Acc}(A^{\{q\}}, i)\right| \cdot \left|\mathrm{Acc}(A_q^{K_s}, n-i)\right| = \Theta\left(\sum_{i=0}^{n} \left|\mathrm{Acc}(A^{K_j}, i)\right| \cdot \left|\mathrm{Acc}(A_q^{K_s}, n-i)\right|\right)$$

(the definition of Θ that is required to hold for all n greater than some n_0 in $\mathbb{N}$ is used here). Thus, by Lemma 2.3.8, we have

$$\begin{aligned}
\#\left[q, \mathrm{Acc}(A^{K_s}, n)\right] &= \sum_{i=0}^{n} \left|\mathrm{Acc}(A^{\{q\}}, i)\right| \cdot \left|\mathrm{Acc}(A_q^{K_s}, n-i)\right| = \\
&= \Theta\left(\sum_{i=0}^{n} \left|\mathrm{Acc}(A^{K_j}, i)\right| \cdot \left|\mathrm{Acc}(A_q^{K_s}, n-i)\right|\right) \\
&= \sum_{i=0}^{n} \Theta\left(i^{r_1-1} \cdot \rho_{max}^i\right) \cdot \Theta\left((n-i)^{r_2-1} \cdot \rho_{max}^{n-i}\right) = \\
&= \Theta\left(\rho_{max}^n \cdot \sum_{i=0}^{n} i^{r_1-1} \cdot (n-i)^{r_2-1}\right) = \Theta\left(n^{r_1+r_2-1} \cdot \rho_{max}^n\right) = \\
&= \Theta\left(n^r \cdot \rho_{max}^n\right).
\end{aligned}$$

(ii) In this case, there has to be a path x_1 in $\text{Pth}[K_0, K_j]$ with r_1 strongly connected components with Perron-Frobenius eigenvalue ρ_{max}, and a path x_2 in $\text{Pth}[K_j, K_s]$ with r_2 strongly connected components with Perron-Frobenius eigenvalue ρ_{max}, so that $r_1 + r_2 = r$.

Then, the result may be proved in a similar manner that in the case of claim (i), with the difference that the case $r_1 = 0$ (resp. $r_2 = 0$) has to be handled as well.

(iii) This claim is considered to be obvious.

The lemma is proved. □

Lemma 2.3.30 Let the symbols have the same meaning as in Notation 2.3.22. Let q be a state in K_j, for some j in $\{0, 1, \ldots, k-1\}$. Let us denote by ρ_j the Perron-Frobenius eigenvalue corresponding to K_j, and by ρ_{max} the greatest of the Perron-Frobenius eigenvalues corresponding to strongly connected components on paths in $\text{Pth}[K_0, K_j, *]$. Let us denote by r the greatest positive integer, such that a path x in $\text{Pth}[K_0, K_j, *]$ exists, such that there are r strongly connected components on x with the Perron-Frobenius eigenvalue equal to ρ_{max}. Then:

(i) If $\rho_{max} > 0$, and $\rho_j = \rho_{max}$, then

$$\#\left[q, \text{Comp}(A, n)\right] = \Theta\left(n^r \cdot \rho_{max}^n\right).$$

(ii) If $\rho_{max} > 0$, and $\rho_j < \rho_{max}$, then

$$\#\left[q, \text{Comp}(A, n)\right] = \Theta\left(n^{r-1} \cdot \rho_{max}^n\right).$$

(iii) If $\rho_{max} = \rho_j = 0$, then

$$\#\left[q, \text{Comp}(A, n)\right] = 0$$

for all n greater than some n_0 in $\mathbb{N}$.

Proof. The lemma is a direct consequence of Lemma 2.3.29, and of the obvious fact that

$$\#\left[q, \text{Comp}(A, n)\right] = \sum_{i=0}^{k-1} \#\left[q, \text{Acc}(A^{K_i}, n)\right],$$

where k is a constant. □

In our report [26], the reader may find a demonstration of Lemma 2.3.30 applied to an example. Now, we shall finally use the theory developed to prove the characterization of the family of weakly state-$\mathcal{C}_=$-equiloaded deterministic finite automata.

Theorem 2.3.31 Let $A = (K, \Sigma, \delta, q_0, F)$ be a DFAε having a connected graphical representation, such that its graphical representation has k strongly connected components, corresponding to disjoint sets of states $K_0, K_1, \ldots, K_{k-1}$, such that

$$\bigcup_{i=0}^{k-1} K_i = K,$$

q_0 is in K_0, and every strongly connected component is reachable.[10] For $i = 0, 1, \ldots, k-1$, let ρ_i denote the Perron-Frobenius eigenvalue of the transition matrix $\Delta_{A[K_i, q_i]}$ of the automaton $A[K_i, q_i]$, where q_i is an arbitrary state in the set K_i, for $i = 1, \ldots, k-1$. Then, the automaton A is weakly state-$\mathcal{C}_=$-equiloaded, if and only if the following properties hold:

[10] It is clearly possible to construct an equivalent DFAε satisfying the property of reachability to every DFAε by simply deleting the strongly connected components that are not reachable.

(i) The Perron-Frobenius eigenvalues corresponding to the strongly connected components of the graphical representation of the automaton A are all the same, i.e.,

$$\rho_0 = \rho_1 = \ldots = \rho_{k-1}.$$

(ii) The length of the longest path x_i in $\mathrm{Pth}[K_0, K_i, *]$, is equal for all i in $\{0, 1, \ldots, k-1\}$, or $\rho_i = 0$ for $i = 0, 1, \ldots, k-1$.

Proof. We shall prove each implication separately.

$\Rightarrow$: First, let us suppose that the property (i) does not hold. Then, indices i, j in $\{0, 1, \ldots, k-1\}$ exist, such that $\rho_i < \rho_j$. Moreover, let ρ_j be the greatest of all Perron-Frobenius eigenvalues. Let q_i in K_i be a state. Let ρ_{max} be a greatest of the Perron-Frobenius eigenvalues corresponding to the strongly connected components on the paths in $\mathrm{Pth}[K_0, K_i, *]$. It follows from Lemma 2.3.30 that

$$\#[q_i, \mathrm{Comp}(A, n)] = \Theta\left(n^r \cdot \rho_{max}^n\right),$$

for some r in $\mathbb{N}$.

Now, if $\rho_{max} < \rho_j$, then, by Lemma 2.3.30, the property

$$\#[q_j, \mathrm{Comp}(A, n)] = \Theta\left(n^{r'} \cdot \rho_j^n\right)$$

holds for some q_j in K_j and r' in $\mathbb{N}$, and the automaton A is obviously not weakly state-$\mathcal{C}_=$-equiloaded.

If $\rho_{max} = \rho_j$, then s in $\{0, 1, \ldots, k-1\}$ has to exist, such that $\rho_s = \rho_j = \rho_{max}$, and by Lemma 2.3.30, such that

$$\#[q_s, \mathrm{Comp}(A, n)] = \Omega\left(n^{r+1} \cdot \rho_{max}^n\right)$$

for some q_s in K_s. It is clear that the automaton A is not weakly state-$\mathcal{C}_=$-equiloaded.

Finally, let us suppose that the property (i) holds, and that the property (ii) does not hold. It clearly follows from Lemma 2.3.30 that the automaton A is not weakly state-$\mathcal{C}_=$-equiloaded.

$\Leftarrow$: This implication follows directly from Lemma 2.3.30.

The characterization is proved. □

Although we leave the characterizations of weakly state-$\mathcal{S}$-equiloaded DFA and DFAε open for $\mathcal{S}$ in $\{\mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$, let us note that the above-presented characterization for $\mathcal{S} = \mathcal{C}_=$ and the methods used in its proof give us a sharp intuition also for the remaining cases. Thus, we may conjecture that a deterministic finite automaton is weakly state-$\mathcal{C}_\leq$-equiloaded if and only if it is weakly state-$\mathcal{C}_=$-equiloaded. Furthermore, we suppose that the characterization of weak state-$\mathcal{A}_=$-equiloadedness will be the same as the characterization of weak state-$\mathcal{C}_=$-equiloadedness, with an additional requirement related to periodicity. Anyway, we leave the details as a subject for the later study.

2.3.6 Relations between the Families of $\mathcal{S}$-Equiloaded Languages

In this subsection, we shall examine the mutual relations between the families of $\mathcal{S}$-equiloaded and weakly $\mathcal{S}$-equiloaded DFA-languages and DFAε-languages. Since the only families of such languages that have been studied up to now are the families $\mathscr{L}_{K-EQ-DFA}(\mathcal{A}_=)$, $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$, and $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_=)$, most of the results presented in this subsection are completely new.

We shall start by presenting the result that shows a certain robustness of the definition of weak transition-$\mathcal{S}$-equiloadedness: we shall prove that the families of weakly transition-$\mathcal{S}$-equiloaded DFA-languages are the same, no matter what $\mathcal{S}$ from the set $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$ is chosen. The same property holds also for the families of weakly transition-$\mathcal{S}$-equiloaded DFAε-languages.

Theorem 2.3.32 The following identities hold:

1. $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_=) = \mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_=) = \mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_\leq) = \mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_\leq)$,
2. $\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_=) = \mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{A}_=) = \mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_\leq) = \mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{A}_\leq)$.

Proof. Let L be a regular language, for which a deterministic finite automaton (with or without ε-transitions) A exists, such that $L(A) = L$ and the graphical representation of the automaton A does not contain any reachable directed cycle from which some accepting state is reachable. Then, clearly, a deterministic finite automaton A' accepting L exists, such that its graphical representation does not contain any reachable directed cycle – it clearly suffices to delete all states q of the automaton A, such that there is not any accepting state reachable from q in A.

The statement of the theorem is then a direct consequence of this fact, of Theorem 2.3.19, and of Theorem 2.3.20. □

Now we shall prove that, similarly as in the case of the strict $\mathcal{S}$-equiloadedness, the use of ε-transitions strengthens the computational power of weakly transition-$\mathcal{S}$-equiloaded deterministic finite automata, as well as transition-$\mathcal{S}$-equiloaded deterministic finite automata.

Theorem 2.3.33 The following strict inclusions hold for all $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$:

1. $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{S}) \subsetneq \mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{S})$,
2. $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{S}) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{S})$.

Proof. According to Theorem 2.3.32, it is sufficient to prove the theorem for one arbitrary $\mathcal{S}$ in the set $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$. We shall prove the theorem for $\mathcal{S} = \mathcal{C}_=$.

Let us consider the language $L = \{a\}^+$. In the proof of Theorem 2.2.8, we have constructed a deterministic finite automaton with ε-transitions accepting L. It can be easily proved that this automaton is transition-$\mathcal{C}_=$-equiloaded. Thus, the language L is both in $\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_=)$ and in $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=)$.

However, on the other hand, the graphical representation of the minimal DFA accepting L (that can be easily constructed by the reader) is not strongly connected and, at the same time, contains a reachable directed cycle. Thus, by Theorem 2.3.21, the language L is not the member of the family $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_=)$ and, as a consequence, nor of the family $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$. That is, the theorem is proved. □

In the following theorem we shall observe that, unlike in the case of weak transition-$\mathcal{S}$-equiloadedness, the families of transition-$\mathcal{S}$-equiloaded DFA(ε)-languages are not all the same for $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$. However, in Theorem 2.3.35, we shall prove that the only difference between these families of languages is in finite languages – that is, these families of languages are *almost* the same.

Theorem 2.3.34 The following relations hold:

1. $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$,
2. $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_=)$,
3. $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$,
4. $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_=)$.
5. The families $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_\leq)$ and $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_\leq)$ are incomparable.
6. The families $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_\leq)$ and $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_\leq)$ are incomparable.

Proof. First, we shall prove that $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$ and $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_=)$. Let L be a finite language. Then, clearly, a deterministic finite automaton with acyclic graphical representation exists, accepting L. However, deterministic finite automata with acyclic graphical representation are clearly both transition-$\mathcal{C}_=$-equiloaded and transition-$\mathcal{A}_=$-equiloaded. Thus, both identities hold for finite languages. Now, let L be an infinite language. Then, the graphical representation of every deterministic finite automaton accepting L has to contain at least one reachable directed cycle from which some accepting state is reachable. However, by Theorem 2.3.17, such an automaton is transition-$\mathcal{C}_=$-equiloaded if and only if it is transition-$\mathcal{A}_=$-equiloaded. Thus, the identities $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$ and $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=) = \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_=)$ are proved.

Now, we shall prove the proper inclusions from claims 1 – 4. First, we shall at once prove the inclusions $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$ and $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=)$. Let us consider the language $L_1 = \{ab\}$. The language L_1 is finite, and thus, clearly is both in $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$, and in $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=)$. We shall prove that there is not any transition-$\mathcal{C}_\leq$-equiloaded DFAε accepting L_1. This will also clearly imply that there is not any transition-$\mathcal{C}_\leq$-equiloaded DFA accepting L_1. For the purpose of contradiction, let us suppose that such a DFAε A exists. Let q be the first state of this automaton (in the direction of every computation path of the automaton A), such that there is a non-ε transition leading from q. Clearly, there has to be a transition $e = (q, a, q')$ on a leading from q to some other state q' – otherwise, there would not be any word in L_1 beginning with a, and that would be a contradiction. Moreover, there has to be a transition f on b reachable from q', such that for every computation path γ in $\mathrm{Comp}(A)$, the inequality $\#[f, \gamma] \leq 1$ holds, and $\#[f, \gamma] = 1$ implies also $\#[e, \gamma] = 1$ – otherwise, the word ab would either not be in L_1, or there would be some other word in L_1. This, togehter with our assumption of $\mathcal{C}_\leq$-equiloadedness, implies also that $\mathrm{Comp}(A)$ is finite. However, there clearly is a computation path γ' in $\mathrm{Comp}(A)$, such that $\#[e, \gamma] \geq 1$ and, at the same time, $\#[f, \gamma] = 0$. Thus, $\#[e, \mathrm{Comp}(A)] > \#[f, \mathrm{Comp}(A)]$. However, since $\mathrm{Comp}(A)$ is finite, there is a nonnegative integer n_0 in $\mathbb{N}$, such that $\mathrm{Comp}(A) = \mathrm{Comp}(A, \leq n)$ for all $n \geq n_0$. Thus, $\#[e, \mathrm{Comp}(A, \leq n)] > \#[f, \mathrm{Comp}(A, \leq n)]$ for all $n \geq n_0$, and that clearly implies that the inequality $B_A(\mathcal{C}_\leq) < 1$ holds. That contradicts our assumption that the automaton A is transition-$\mathcal{C}_\leq$-equiloaded.

Now, we shall simultaneously prove the inclusions $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$, and $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_\leq) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=)$. Let us consider the language $L_2 = \{a, b, aa, ba, cc, dc\}$. Since the language L_2 is finite, it is clearly both in $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$, and in $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=)$. Again, we shall prove that there is not any transition-$\mathcal{A}_\leq$-equiloaded DFAε accepting L_2, and this will imply the same consequence also for DFA. For the purpose of contradiction, let us suppose that a transition-$\mathcal{A}_\leq$-equiloaded DFAε A exists, such that $L(A) = L_2$. Let q be the first (in the direction of every computation path of the automaton) state of the automaton A, such that there is at least one non-ε transition leading from A. Then, there has to be a transition e_x on each character x in the set $\{a, b, c, d\}$ leading from q – otherwise, the property $L(A) = L_2$ would not be satisfied. Moreover, it is clear that the property $(q, w) \vdash^+ (q, \varepsilon)$ cannot hold for any w in Σ^+ – otherwise the language $L(A)$ would be either empty or infinite, and that would contradict $L(A) = L_2$. Thus, each of these transitions is used exactly once in accepting computations on words beginning with the corresponding character, and is not used in other accepting computations. Thus, for each x in $\{a, b, c, d\}$,

$$\#[x, \mathrm{Acc}(A)] = |\{w \in L_2 \mid w[1] = x\}|.$$

However, there are two words in L_2 beginning with a and b, but only one word in L beginning with c and d. Thus,

$$\#[a, \mathrm{Acc}(A)] > \#[c, \mathrm{Acc}(A)],$$

and since the accepted language L_2 is finite, for n greater than some n_0 in $\mathbb{N}$, we have also

$$\#[a, \mathrm{Acc}(A, \leq n)] > \#[c, \mathrm{Acc}(A, \leq n)].$$

That is, $B_A(\mathcal{A}_\leq) < 1$, and that contradicts our assumption that the automaton A is transition-$\mathcal{A}_\leq$-equiloaded.

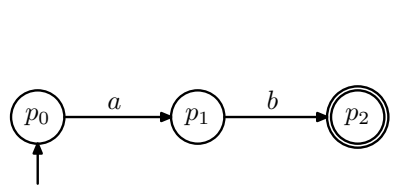
It remains to prove the claims from points 5 and 6. These claims will be obviously proved, if we construct a transition-$\mathcal{A}_{\leq}$-equiloaded DFA (that can be obviously viewed also as a DFAε) $A_1 = (K_1, \Sigma_1, \delta_1, p_0, F_1)$, such that $L(A_1) = L_1$, and a transition-$\mathcal{C}_{\leq}$-equiloaded DFA $A_2 = (K_2, \Sigma_2, \delta_2, q_0, F_2)$, such that $L(A_2) = L_2$. We shall define these automata as follows: $K_1 = \{p_0, p_1, p_2\}$, $\Sigma_1 = \{a, b\}$, $F = \{p_2\}$, and

$$\delta_1(p_0, a) = p_1,$$
$$\delta_1(p_1, b) = p_2.$$

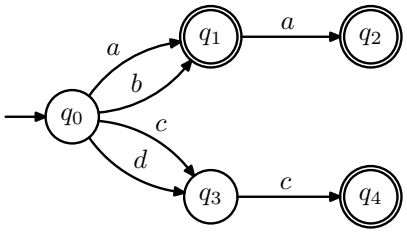
As it can be easily observed,

$$\#[e, \mathrm{Acc}(A, \leq n)] = 1$$

for every transition e in D_{A_1} and all n in $\mathbb{N}$, such that $n \geq 2$. Thus, the automaton A_1 is transition-$\mathcal{A}_{\leq}$-equiloaded.



(a) The automaton A_1.



(b) The automaton A_2.

Figure 2.3: The transition-$\mathcal{A}_{\leq}$-equiloaded automaton A_1 accepting the language L_1 and the transition-$\mathcal{C}_{\leq}$-equiloaded automaton A_2 accepting the language L_2.

For the automaton A_2, we shall define $K_2 = \{q_0, q_1, q_2, q_3, q_4\}$, $\Sigma = \{a, b, c, d\}$, $F = \{q_1, q_2, q_4\}$, and

$$\delta_2(q_0, a) = q_1, \quad \delta_2(q_0, b) = q_1, \quad \delta_2(q_0, c) = q_3,$$
$$\delta_2(q_0, d) = q_3, \quad \delta_2(q_1, a) = q_2, \quad \delta_2(q_3, c) = q_4.$$

The reader may easily convince himself that $L(A_2) = L_2$. Moreover, as can be clearly observed,

$$\#[e, \mathrm{Comp}(A, \leq n)] = 2$$

for every transition e in D_{A_2} and all n in $\mathbb{N}$, such that $n \geq 2$. Thus, the automaton A_2 is transition-$\mathcal{C}_{\leq}$-equiloaded. That is, the theorem is proved. □

Theorem 2.3.35 For a given family of languages $\mathscr{L}$, let $\mathscr{L}^{inf}$ denote the family of all infinite languages in $\mathscr{L}$. Then, the following identities hold:

1. $\mathscr{L}^{inf}_{\delta-EQ-DFA}(\mathcal{C}_{=}) = \mathscr{L}^{inf}_{\delta-EQ-DFA}(\mathcal{A}_{=}) = \mathscr{L}^{inf}_{\delta-EQ-DFA}(\mathcal{C}_{\leq}) = \mathscr{L}^{inf}_{\delta-EQ-DFA}(\mathcal{A}_{\leq})$,
2. $\mathscr{L}^{inf}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_{=}) = \mathscr{L}^{inf}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_{=}) = \mathscr{L}^{inf}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_{\leq}) = \mathscr{L}^{inf}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_{\leq})$.

Proof. Let L in $\mathscr{R}^{inf}$ be an infinite regular language. Then, since L is infinite, the graphical representation of every deterministic finite automaton accepting L has to contain at least one reachable directed cycle from which some accepting state is reachable. Thus, both claims are direct consequences of Theorem 2.3.17. □

We shall omit the proof of the following theorem, since it is a direct generalization of a theorem that we have presented in [25]. The proof may be found in our report [26].

Theorem 2.3.36 For every $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$, the strict inclusion

$$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{S}) \subsetneq \mathscr{L}_{\delta-WEQ-DFA}(\mathcal{S})$$

holds.

We leave an analogous property for the families of $\mathcal{S}$-equiloaded resp. weakly $\mathcal{S}$-equiloaded DFAε-languages as an open problem.

Open Problem 2.3.37 The inclusion $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{S}) \subseteq \mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{S})$ holds trivially for all $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$. Is this inclusion proper or does $\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{S}) = \mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{S})$ hold for $\mathcal{S}$ in the set $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$?

Now, let us consider the families of state-$\mathcal{S}$-equiloaded and weakly state-$\mathcal{S}$-equiloaded languages. Let us note that a theorem analogous to Theorem 2.3.33 holds also for the families of (weakly) state-$\mathcal{S}$-equiloaded languages. Since exactly the same counterexample $L = \{a\}^+$ may be used to prove the theorem,[11] we shall omit the proof.

Theorem 2.3.38 The following strict inclusions hold for all $\mathcal{S}$ in $\{\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq\}$:

1. $\mathscr{L}_{K-WEQ-DFA}(\mathcal{S}) \subsetneq \mathscr{L}_{K-WEQ-DFA\varepsilon}(\mathcal{S})$,
2. $\mathscr{L}_{K-EQ-DFA}(\mathcal{S}) \subsetneq \mathscr{L}_{K-EQ-DFA\varepsilon}(\mathcal{S})$.

We shall leave open the relations between the families of (weakly) state-$\mathcal{S}$-equiloaded languages for different choices of $\mathcal{S}$.

Finally in this subsection, let us consider the relations between $\mathcal{S}$-equiloaded and strictly $\mathcal{S}$-equiloaded DFA(ε)-languages, for $\mathcal{S} = \mathcal{C}_=$ and $\mathcal{S} = \mathcal{A}_=$.

Theorem 2.3.39 The following strict inclusions hold:

1. $\mathscr{L}_{K-SEQ-DFA}(\mathcal{C}) \subsetneq \mathscr{L}_{K-EQ-DFA}(\mathcal{C}_=)$,
2. $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{C}) \subsetneq \mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$,
3. $\mathscr{L}_{K-SEQ-DFA}(\mathcal{A}) \subsetneq \mathscr{L}_{K-EQ-DFA}(\mathcal{A}_=)$,
4. $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{A}) \subsetneq \mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$,
5. $\mathscr{L}_{K-SEQ-DFA\varepsilon}(\mathcal{C}) \subsetneq \mathscr{L}_{K-EQ-DFA\varepsilon}(\mathcal{C}_=)$,
6. $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}(\mathcal{C}) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_=)$,
7. $\mathscr{L}_{K-SEQ-DFA\varepsilon}(\mathcal{A}) \subsetneq \mathscr{L}_{K-EQ-DFA\varepsilon}(\mathcal{A}_=)$,
8. $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}(\mathcal{A}) \subsetneq \mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_=)$.

Proof. The inclusions are a direct consequence of Theorem 1.6.2. Let us prove that these inclusions are proper. Let us consider the languages $L_1 = \{a\}^* \cdot \{b\}^*$ and $L_2 = \{a, b\}^*$. It is an easy exercise to prove that L_1 is not in $\mathscr{L}_{K-SEQ-DFA}(\mathcal{C})$, nor in $\mathscr{L}_{K-SEQ-DFA}(\mathcal{A})$ and that it is in $\mathscr{L}_{K-EQ-DFA}(\mathcal{C}_=)$, as well as in $\mathscr{L}_{K-EQ-DFA}(\mathcal{A}_=)$. Similarly, it is easy to prove that L_2 is not in $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{C})$, nor in $\mathscr{L}_{\delta-SEQ-DFA}(\mathcal{A})$, and that it is both in $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_=)$, and in $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_=)$. □

2.3.7 Closure Properties

In this subsection, we shall briefly state the known closure properties of the families of $\mathcal{S}$-equiloaded and weakly $\mathcal{S}$-equiloaded languages. We shall omit proofs – the proofs of older results may be found in the theses [27], [28] and [25], and the proofs of newer results may be found in our report [26].

For the case of the families of state-$\mathcal{S}$-equiloaded and weakly state-$\mathcal{S}$-equiloaded DFA(ε)-languages, we leave most of their closure properties open. In the following theorem, we shall only restate the closure properties of the family $\mathscr{L}_{K-EQ-DFA}(\mathcal{A}_=)$, proved already in [27] and

[11] It is easy to prove that this language is not in $\mathscr{L}_{K-WEQ-DFA}(\mathcal{S})$ by using the Myhill-Nerode theorem [33].

[28]. The main reason for this is our belief that these closure properties will be less painfully provable after slightly extending the theory from the previous subsection (it will be possible to prove more closure properties at once, and thus the need to examine each case separately will be eliminated).

Theorem 2.3.40 The family $\mathscr{L}_{K-EQ-DFA}(\mathcal{A}_{=})$ is not closed under concatenation, union, intersection, complementation, reversal, homomorphism, and inverse homomorphism.

In the following theorem, we shall summarize the closure properties of the families of transition-$\mathcal{S}$-equiloaded and weakly transition-$\mathcal{S}$-equiloaded DFA(ε)-languages. The results obtained in the previous subsection enable us to prove the closure properties for relatively large numbers of families of languages at once (for details, see our report [26]). Thus, our knowledge of the closure properties is better than in the case of state-$\mathcal{S}$-equiloaded and weakly state-$\mathcal{S}$-equiloaded languages.

Theorem 2.3.41 The closure properties of the families of transition-$\mathcal{S}$-equiloaded and weakly transition-$\mathcal{S}$-equiloaded DFA(ε)-languages can be summarized by Table 2.2.

	$\cdot$	$\cup$	$\cap$	C	$*$	$+$	R	h	h^{-1}
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{A}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{A}_{\leq})$	No	No	No	No	?	?	?	?	No

Table 2.2: The closure properties of the families of transition-$\mathcal{S}$-equiloaded and weakly transition-$\mathcal{S}$-equiloaded DFA(ε)-languages.

Chapter 3

Strictly $\mathcal{S}$-Equiloaded Deterministic One-Counter Automata

In this chapter, we shall study the families of strictly $\mathcal{S}$-equiloaded deterministic one-counter automata and the corresponding families of languages, for $\mathcal{S} = \mathcal{C}$, $\mathcal{S} = \mathcal{A}$, and $\mathcal{S} = \mathcal{E}$. Equiloaded one-counter automata have not been studied yet. That is, all results presented in this chapter are new.

3.1 Examples of Strictly $\mathcal{S}$-Equiloaded DOCA-Languages

In order to build an adequate picture of the computational power of strictly $\mathcal{S}$-equiloaded one-counter automata, we shall work out several examples.

Example 3.1.1 In this example, we shall present three examples of languages in the families $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$, and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$. Let us consider the following three regular languages: $L_1 = \{a\}^*\{b\}^*$, $L_2 = \{a\}^*\{\varepsilon, b\}$, and $L_3 = \{a\}^* \cup \{b\}$. For each of these languages, we shall construct a deterministic one-counter automaton that is both strictly state-$\mathcal{C}$-equiloaded, and strictly state-$\mathcal{A}$-equiloaded.

We shall define a deterministic one-counter automaton $A_1 = (K_1, \Sigma_1, \delta_1, q_{0,1}, F_1)$ accepting the language L_1 as follows: $K_1 = \{q_0\}$, $\Sigma_1 = \{a, b\}$, $q_{0,1} = q_0$, $F_1 = \{q_0\}$, and

$$\delta_1(q_0, a, 0) = (q_0, 0), \qquad \delta_1(q_0, b, 0) = (q_0, +1), \qquad \delta_1(q_0, b, 1) = (q_0, 0).$$

The claim $L(A_1) = L_1$ is considered to be obvious. The strict state-$\mathcal{S}$-equiloadedness of the automaton A_1 for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$ is a direct consequence of the fact that the automaton has only one state.

Now, let us construct a deterministic one-counter automaton $A_2 = (K_2, \Sigma_2, \delta_2, q_{0,2}, F_2)$, such that $L(A_2) = L_2$. We shall define the automaton as follows: $K_2 = \{q_0\}$, $\Sigma_2 = \{a, b\}$, $q_{0,2} = q_0$, $F_2 = \{q_0\}$, and

$$\delta_2(q_0, a, 0) = (q_0, 0), \qquad \delta_2(q_0, b, 0) = (q_1, +1).$$

Once again, the claim $L(A_2) = L_2$ is considered to be obvious, and the strict state-$\mathcal{S}$-equiloadedness of the automaton A_2 for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$ follows from the fact that there is only one state in K_2.

Finally, we shall construct a deterministic one-counter automaton $A_3 = (K_3, \Sigma_3, \delta_3, q_{0,3}, F_3)$ accepting the language L_3. We shall define the automaton as follows: $K_3 = \{q_0, q_1\}$, $\Sigma_3 = \{a, b\}$, $q_{0,3} = q_0$, $F_3 = \{q_0, q_1\}$, and

$$\begin{aligned} &\delta_3(q_0, a, 0) = (q_1, +1), \qquad &&\delta_3(q_0, b, 0) = (q_1, 0), \\ &\delta_3(q_0, a, 1) = (q_1, 0), \qquad &&\delta_3(q_1, a, 1) = (q_0, 0). \end{aligned}$$

The claim $L(A_3) = L_3$ is considered to be obvious. The strict state-$\mathcal{S}$-equiloadedness of the automaton A_3 for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$ follows from the fact that every computation path γ of the automaton A_3 alternates between states q_0 and q_1. Thus, as a consequence we have

$$|\#[q_0, \gamma] - \#[q_1, \gamma]| \leq 1$$

for every computation path γ, and the automaton A_3 is strictly state-$\mathcal{C}$-equiloaded, as well as strictly state-$\mathcal{A}$-equiloaded.

In the construction of each of the three automata from the previous example, we have exploited the fact that we can use one state of a deterministic one-counter automaton to act like two states of a deterministic finite automaton – if we use only two values of the counter, 0 and 1, then we may view the deterministic one-counter automaton with the set of states K as a deterministic finite automaton with the set of states $K \times \{0, 1\}$. However, the differences between the number of uses of a state q with the counter value 0 and the number of uses of the state q with the counter value 1 have no effect on the strict equiloadedness: the only thing we care about is the overall number of uses of the state q (without taking the counter value into consideration). Thus, the families of strictly state-$\mathcal{S}$-equiloaded DOCA-languages contain also some regular languages that are not strictly state-$\mathcal{S}$-equiloaded DFAε-languages (however, as we shall observe later, strictly state-$\mathcal{S}$-equiloaded DOCA may accept also nonregular languages).

This method cannot be used for strict transition-$\mathcal{S}$-equiloadedness. Unlike in the case of states, the transition from a state q on some c in $\Sigma \cup \{\varepsilon\}$ is a different transition for a counter value 0, and for a counter value greater than 0. However, there are other methods that can be used in the construction of strictly transition-$\mathcal{S}$-equiloaded DOCA for languages that are not in $\mathscr{L}_{\delta-SEQ-DFA}$ and $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$. We shall present one such method in the following example.

Example 3.1.2 Let us consider the language $L = \{ab, ba\}^*$. We shall observe that this language is both in $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C})$, and in $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A})$. The main idea of the construction of a strictly transition-$\mathcal{S}$-equiloaded DOCA $A = (K, \Sigma, \delta, q_0, F)$ accepting L is as follows: each computation of the automaton A is required to begin with a sequence of ε-transitions that increases the counter from 0 to 5. Afterwards, the computation is required to "decide" (in the state q_5) between two cycles: after passing the first of the cycles, the character a is read, and the counter is decreased by the value of 2 (if the counter value is less than 2 at the beginning of the cycle, the computation gets stuck during the cycle). After passing the second cycle, the character b is read, and the counter is decreased by the value of 3. The computation is allowed to proceed from the state q_5 to the initial state only if the counter value 0 is reached. This is clearly the case if and only if each of the cycles is passed exactly once.

Formally, we shall construct the deterministic one-counter automaton A as follows: we shall define $K = \{q_0, q_1, \ldots, q_8\}$, $\Sigma = \{a, b\}$, $F = \{q_0\}$, and

$$\begin{array}{llll} \delta(q_0, \varepsilon, 0) = (q_1, +1), & \delta(q_1, \varepsilon, 1) = (q_2, +1), & \delta(q_2, \varepsilon, 1) = (q_3, +1), & \delta(q_3, \varepsilon, 1) = (q_4, +1), \\ \delta(q_4, \varepsilon, 1) = (q_5, +1), & \delta(q_5, \varepsilon, 0) = (q_0, 0), & \delta(q_5, a, 1) = (q_6, -1), & \delta(q_5, b, 1) = (q_7, -1), \\ \delta(q_6, \varepsilon, 1) = (q_5, -1), & \delta(q_7, \varepsilon, 1) = (q_8, -1), & \delta(q_8, \varepsilon, 1) = (q_5, -1). & \end{array}$$

The reader may easily convince himself that $L(A) = L$. The automaton is strictly transition-$\mathcal{S}$-equiloaded for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$, since, for the above explained reasons, the property

$$|\#[e, \gamma] - \#[f, \gamma]| \leq 3$$

has to hold for all e, f in D, and for every computation path γ (for accepting computation paths, the constant 3 may be replaced by 0).

Example 3.1.3 Let us consider the deterministic one-counter automaton A from the previous example, once again. Clearly,

$$N(A) = \{ab, ba\}^*\{\varepsilon, aaa, aab, bb\} =: L'.$$

We have already noted that the automaton A is strictly transition-$\mathcal{C}$-equiloaded, and thus also strictly transition-$\mathcal{E}$-equiloaded. Thus, the language L' is both in $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$, and in $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E})$.

In the final three examples of this section, we shall return to strictly state-$\mathcal{S}$-equiloaded DOCA-languages. We shall show that among these languages, there are also some nonregular languages. Moreover, we shall observe that in the families of languages $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$, there are also some languages that are not prefix-dense.

Example 3.1.4 In this example, we shall show that the (obviously nonregular) language

$$L = \{a^n b^n \mid n \geq 0\}$$

is in $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$. Let us define a deterministic one-counter automaton $A = (K, \Sigma, \delta, q_0, F)$ as follows: $K = \{q_0, q_1\}$, $\Sigma = \{a, b\}$, $F = \emptyset$ (since we are interested in the language accepted by empty memory, the set F is irrelevant), and

$$\begin{aligned} &\delta(q_0, a, 0) = (q_0, +1), \qquad \delta(q_0, a, 1) = (q_0, +1), \\ &\delta(q_0, b, 1) = (q_1, -1), \qquad \delta(q_1, b, 1) = (q_1, -1). \end{aligned}$$

We consider the claim $L(A) = L$ to be obvious. Moreover, in every computation path γ of the automaton A, accepting by empty memory, the following property clearly holds:

$$\#[q_1, \gamma] = \#[q_0, \gamma] - 1.$$

Thus, the automaton A is strictly state-$\mathcal{E}$-equiloaded. As a consequence, the language L is in $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$.

As shown in Example 1.7.6, the language L is not prefix-dense. Thus, it is neither in the family $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$, nor in the family $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$.

Example 3.1.5 In this example, we shall observe that also the families $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$ contain some nonregular languages. Let us consider the language

$$L = \{\, w \in \{a, b\}^* \mid \forall u \in \{a, b\}^*, u \text{ is a prefix of } w : \#_a(u) \geq \#_b(u) \}\,.$$

The language L is clearly nonregular. However, let us consider a deterministic one-counter automaton $A = (K, \Sigma, \delta, q_0, F)$, defined as follows: $K = \{q_0\}$, $\Sigma = \{a, b\}$, $F = \{q_0\}$, and

$$\delta(q_0, a, 0) = (q_0, +1), \qquad \delta(q_0, a, 1) = (q_0, +1), \qquad \delta(q_0, b, 1) = (q_0, -1).$$

Clearly, $L(A) = L$. The strict state-$\mathcal{S}$-equiloadedness of the automaton A, for both $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$, follows from the fact that the automaton A has only one state. Thus, L is both in the family $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$, and in the family $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$.

Example 3.1.6 In this final example, we shall show that also the family $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ contains some non-prefix-dense language. Let us consider the language

$$L = \{a^n b \mid n \geq 1\} \cup \{\varepsilon\}.$$

This language clearly is not prefix-dense. Let us consider the following deterministic one-counter automaton: $A = (K, \Sigma, \delta, q_0, F)$, where $K = \{q_0, q_1\}$, $\Sigma = \{a, b\}$, $F = \emptyset$, and

$$\begin{aligned} &\delta(q_0, a, 0) = (q_0, +1), \qquad \delta(q_0, \varepsilon, 1) = (q_1, 0), \\ &\delta(q_1, a, 1) = (q_0, 0), \qquad \delta(q_1, b, 1) = (q_1, -1). \end{aligned}$$

The statement $N(A) = L$, as well as the strict state-$\mathcal{S}$-equiloadedness of the automaton A for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}\}$, are considered to be clear. Thus, L is both in the family $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$, and in the family $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$.

3.2 Lemmas

In this section, we shall prove some auxiliary results that we shall use later on in our study of strictly $\mathcal{S}$-equiloaded deterministic one-counter automata.

Lemma 3.2.1 Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic one-counter automaton. Let an upper bound exist for the length of computation paths (accepting computation paths) [computation paths accepting by empty memory] γ of the automaton A, such that γ reaches a configuration with the state q_0 and with the counter value 0 only at its beginning. Let H_1^* (H_2^*) [H_3^*] be the smallest of such upper bounds. Then,

$$H_1^* \leq |K|(|K|+1) \qquad (H_2^* \leq |K|(|K|+1)) \qquad [H_3^* \leq |K|(|K|+1)].$$

Proof. We shall prove all three claims at once. For the purpose of contradiction, let us suppose that $H_1^* > |K|(|K|+1)$ (that $H_2^* > |K|(|K|+1)$) [that $H_3^* > |K|(|K|+1)$]. Then, a computation path γ_1 (an accepting computation path γ_2) [a computation path γ_3 accepting by empty memory] exists, such that a configuration with the state q_0 and with the counter value 0 is reached only at its beginning, and such that $|\gamma_1| = H_1^*$ ($|\gamma_2| = H_2^*$) [$|\gamma_3| = H_3^*$].

First, let us suppose that the greatest counter value achieved by the computation path γ_1 (the computation path γ_2) [the computation path γ_3] is $t \leq |K|$. Since, by our assumption, the length of the computation path is greater than $|K|(|K|+1)$, it follows by the Pigeonhole principle that the computation path reaches at least two configurations with the same state and the same counter value. That is, the computation path γ_1 (the accepting computation path γ_2) [the computation path γ_3 accepting by empty memory] corresponds to some computation (some accepting computation) [some computation accepting by empty memory] of the form

$$(q_0, uvw, 0) \vdash^* (q, vw, t) \vdash^+ (q, w, t) \vdash^* (q', \varepsilon, t'),$$

where u, v, and w in Σ^* are words, t and t' in $\mathbb{N}$ are nonnegative integer counter values, and q and q' in K are states. Moreover, for the case of the computation path γ_2, the state q' is accepting (in F), and for the case of the computation path γ_3, the counter value t' is zero.

It is clear that there is a computation (an accepting computation) [a computation accepting by empty memory] on the word uv^2w of the form

$$(q_0, uv^2w, 0) \vdash^* (q, v^2w, t) \vdash^+ (q, vw, t) \vdash^+ (q, w, t) \vdash^* (q', \varepsilon, t').$$

Clearly, the length of the corresponding computation path (accepting computation path) [computation path accepting by empty memory] is greater than the length of the computation path γ_1 (the computation path γ_2) [the computation path γ_3]. However, this computation path clearly reaches a configuration with the state q_0 and with the counter value 0 only at its beginning, and thus, H_1^* (H_2^*) [H_3^*] is not an upper bound for the length of the corresponding set of computation paths. That is a contradiction.

Now, let us suppose that the greatest counter value achieved by the computation path γ_1 (the computation path γ_2) [the computation path γ_3] is $t \geq |K|+1$. Then, it follows from the Pigeonhole principle that the computation path γ_1 (the computation path γ_2) [the computation path γ_3] corresponds to some computation (to some accepting computation) [to some computation accepting by empty memory] of the form

$$(q_0, uvwx, 0) \vdash^* (q, vwx, t_1) \vdash^+ (q, wx, t_2) \vdash^* (p, x, t) \vdash^* (p', \varepsilon, t'), \tag{3.1}$$

where u, v, w, and x are words in Σ^*, p, q, and p' in K are states, and t_1, t_2 and t' are nonnegative integer counter values in $\mathbb{N}$, such that $0 < t_1 < t_2$. Moreover, the counter value is always positive during the computation

$$(q, vwx, t_1) \vdash^+ (q, wx, t_2) \vdash^* (p, x, t).$$

For the case of the computation path γ_2, the state p' is in F, and for the case of the computation path γ_3, the counter value t' is zero.

Now, let us first suppose that the counter value is always positive during the computation

$$(p, x, t) \vdash^* (p', \varepsilon, t').$$

(Let us note that this may be the case only for the computation path γ_1 or for the computation path γ_2, not for the computation path γ_3.) It directly follows that

$$\begin{aligned}(q_0, uv^2wx, 0) \vdash^* (q, v^2wx, t_1) \vdash^+ (q, vwx, t_2) \vdash^+ (q, wx, 2t_2 - t_1) \vdash^* \\ \vdash^* (p, x, t + t_2 - t_1) \vdash^* (p', \varepsilon, t' + t_2 - t_1).\end{aligned}$$

The corresponding computation path clearly reaches the configuration with the state q_0 and with the counter value 0 only at its beginning. Moreover, it is longer than the computation path γ_1 (the computation path γ_2). Thus, H_1^* (H_2^*) is not an upper bound of the corresponding lengths of computation paths. That is a contradiction.

Let us now consider the remaining case when the counter value zero is reached during the computation

$$(p, x, t) \vdash^* (p', \varepsilon, t').$$

Then, it follows from the Pigeonhole principle that words x_1, x_2, x_3 in Σ^* exist, such that $x = x_1x_2x_3$, and such that the computation (3.1) can be rewritten as

$$\begin{aligned}(q_0, uvwx_1x_2x_3, 0) \vdash^* (q, vwx_1x_2x_3, t_1) \vdash^+ (q, wx_1x_2x_3, t_2) \vdash^* (p, x_1x_2x_3, t) \vdash^* \\ \vdash^* (q', x_2x_3, t_2') \vdash^+ (q', x_3, t_1') \vdash^* (p', \varepsilon, t'),\end{aligned}$$

for some state q' in K and some nonnegative integer counter values t_1', t_2' in $\mathbb{N}$, such that $0 < t_1' < t_2'$, so that the counter value is always positive during the computation

$$(p, x_1x_2x_3, t) \vdash^* (q', x_2x_3, t_2') \vdash^+ (q', x_3, t_1').$$

Now, let us denote by L the least common multiple of positive integers $t_2 - t_1$ and $t_2' - t_1'$. Let r, s in $\mathbb{N}^+$ be positive integers, such that $(r-1)(t_2 - t_1) = (s-1)(t_2' - t_1') = L$. Then, it is obvious that

$$\begin{aligned}(q_0, uv^rwx_1x_2^sx_3, 0) &\vdash^* (q, v^rwx_1x_2^sx_3, t_1) \vdash^+ (q, v^{r-1}wx_1x_2^sx_3, t_2) \vdash^+ \ldots \vdash^+ \\ &\vdash^+ (q, vwx_1x_2^sx_3, t_2 + (r-2)(t_2 - t_1)) \vdash^+ (q, wx_1x_2^sx_3, t_2 + (r-1)(t_2 - t_1)) \vdash^* \\ &\vdash^* (p, x_1x_2^sx_3, t + (r-1)(t_2 - t_1)) \vdash^* (q', x_2^sx_3, t_2' + (r-1)(t_2 - t_1)) \vdash^+ \\ &\vdash^+ (q', x_2^{s-1}x_3, t_1' + (r-1)(t_2 - t_1)) \vdash^+ \ldots \vdash^+ \\ &\vdash^+ (q', x_2x_3, t_1' + (r-1)(t_2 - t_1) - (s-2)(t_2' - t_1')) \vdash^+ \\ &\vdash^+ (q', x_3, t_1' + (r-1)(t_2 - t_1) - (s-1)(t_2' - t_1')) \vdash^0 (q', x_3, t_1') \vdash^* (p', \varepsilon, t').\end{aligned}$$

That is, $r-1$ loops from the state q, each increasing the counter value by $t_2 - t_1$, and $s-1$ loops from the state q', each decreasing the counter value by $t_2' - t_1'$, can be added to the computation, without changing any of its important properties.

Clearly, the computation path corresponding to this computation reaches a configuration with the state q_0 and with the counter value 0 only at its beginning. Moreover, it is longer than the computation path γ_1 (the computation path γ_2) [the computation path γ_3].

Thus, H_1^* (H_2^*) [H_3^*] is not an upper bound of the corresponding lengths of computation paths. Once again, this is a contradiction. The lemma is proved. □

We shall omit the slightly technical proofs of the following two lemmas. The proofs may be found in our report [26].

Lemma 3.2.2 Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic one-counter automaton. Let q, q' in K be states and t in $\mathbb{N}$ be a nonnegative integer counter value. If the state q' is reachable from some configuration with the state q and with the counter value t, then a computation

$$(q, w, t) \vdash^* (q', \varepsilon, t')$$

exists for some w in Σ^* and t' in $\mathbb{N}$, such that the counter value does not exceed the value

$$T = t + |K|^3$$

during the computation.[1]

Lemma 3.2.3 Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic one-counter automaton. Let p, q in K be states, and s, t in $\mathbb{N}$ be nonnegative integer counter values. If, for some u in Σ^*, $(p, u, s) \vdash^* (q, \varepsilon, t)$, then a nonnegative integer n in $\mathbb{N}$ exists, such that

$$n \leq \max\{s, t\} \cdot |K| + |K|^4 + |K| - 1,$$

and $(p, v, s) \vdash^n (q, \varepsilon, t)$ for some word v in Σ^*.

Corollary 3.2.4 Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic one-counter automaton, let q in K be a state, and t in $\mathbb{N}$ be a nonnegative integer counter value.

1. If some configuration with the state q_0 and the counter value 0 is reachable from some configuration with the state q and the counter value t, then it is reachable in at most

   $$(t + 1) \cdot |K| + |K|^4 - 1$$

   computation steps.

2. If some accepting configuration is reachable from some configuration with the state q and the counter value t, then it is reachable in at most

   $$(t + 1) \cdot |K| + 2 \cdot |K|^4 - 1$$

   computation steps.

3. If some configuration with the counter value 0 is reachable from some configuration with the state q and the counter value t, then it is reachable in at most

   $$(t + 1) \cdot |K| + |K|^4 - 1$$

   computation steps.

Proof. The claims 1 and 3 are direct corollaries of Lemma 3.2.3. The claim 2 follows from Lemma 3.2.2 and Lemma 3.2.3. □

3.3 Characterization of Strict Transition-$\mathcal{S}$-Equiloadedness

In this section, we shall prove the characterization of strictly transition-$\mathcal{S}$-equiloaded deterministic one-counter automata, for $\mathcal{S} = \mathcal{C}$, $\mathcal{S} = \mathcal{A}$, and $\mathcal{S} = \mathcal{E}$. We shall prove this characterization in Theorem 3.3.1. Later in this section, we shall prove that this characterization is decidable and present an algorithm for deciding if a given deterministic one-counter automaton is strictly transition-$\mathcal{S}$-equiloaded.

[1] This upper bound is not meant to be tight.

Theorem 3.3.1 Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic one-counter automaton.

a) A is strictly transition-$\mathcal{C}$-equiloaded, if and only if the following two properties hold:

   (i) For every computation path γ, such that $|\gamma| \geq 1$ and γ reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise, the property
   $$\#[e, \gamma] = 1$$
   holds for every transition e in D.

   (ii) If γ is a computation path of the automaton A, such that γ reaches a configuration with the state q_0 and the counter value 0 only at its beginning, then $|\gamma| \leq M$ for some fixed constant M in $\mathbb{N}$.

b) A is strictly transition-$\mathcal{A}$-equiloaded, if and only if $L(A)$ is empty or if the following two properties hold:

   (i) For every computation path γ, such that $|\gamma| \geq 1$ and γ reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise, the property
   $$\#[e, \gamma] = 1$$
   holds for every transition e in D.

   (ii') If γ is an accepting computation path of the automaton A, such that γ reaches a configuration with the state q_0 and the counter value 0 only at its beginning, then $|\gamma| \leq M$ for some fixed constant M in $\mathbb{N}$.

c) A is strictly transition-$\mathcal{E}$-equiloaded, if and only if the following two properties hold:

   (i) For every computation path γ, such that $|\gamma| \geq 1$ and γ reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise, the property
   $$\#[e, \gamma] = 1$$
   holds for every transition e in D.

   (ii'') If γ is a computation path of the automaton A accepting by empty memory, such that γ reaches a configuration with the state q_0 and the counter value 0 only at its beginning, then $|\gamma| \leq M$ for some fixed constant M in $\mathbb{N}$.

Proof. We shall prove all three claims at once. First, let us prove the left-to-right implications. For the purpose of contradiction, let us suppose that the automaton A is strictly transition-$\mathcal{S}$-equiloaded for some $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$, and that the property (i) does not hold. Moreover, if $\mathcal{S} = \mathcal{A}$, let us further suppose that $L(A)$ is nonempty.

Since (i) does not hold, a computation path γ exists, such that γ corresponds to a computation

$$(q_0, w, 0) \vdash^n (q_0, \varepsilon, 0)$$

for some word w in Σ^* and some positive integer n in $\mathbb{N}^+$, and

$$|\#[e, \gamma] - \#[f, \gamma]| \geq 1$$

for some e, f in D (this is the case because every transition leading from the state q_0 at the counter value 0 can be used at most once in computation paths that we are concerned with in (i)). That is, for all k in $\mathbb{N}$, a computation path γ_k exists, such that γ_k corresponds to a computation

$$(q_0, w^k, 0) \vdash^{kn} (q_0, \varepsilon, 0),$$

and

$$|\#[e, \gamma_k] - \#[f, \gamma_k]| \geq k. \tag{3.2}$$

In the case $\mathcal{S} = \mathcal{C}$, (3.3) clearly contradicts an assumption of strict transition-$\mathcal{C}$-equiloadedness of the automaton A. Moreover, γ_k is clearly a computation path accepting by empty memory for all k in $\mathbb{N}$. Thus, (3.3) clearly leads to a contradiction also in the case $\mathcal{S} = \mathcal{E}$. Finally, in the case $\mathcal{S} = \mathcal{A}$, there has to be a nonnegative integer *constant* r, such that

$$(q_0, u, 0) \vdash^r (q, \varepsilon, t)$$

for some word u in Σ^*, some *accepting* state q in F, and some nonnegative integer t in $\mathbb{N}$. Thus, for an *accepting* computation path γ'_k, corresponding to the computation

$$(q_0, w^k u, 0) \vdash^{kn} (q_0, u, 0) \vdash^r (q, \varepsilon, t),$$

the property

$$\left|\#[e, \gamma'_k] - \#[f, \gamma'_k]\right| \geq k - r \tag{3.3}$$

holds. This clearly contradicts the assumption that the automaton A is strictly transition-$\mathcal{A}$-equiloaded.

Now, let us suppose that $\mathcal{S} = \mathcal{C}$ and the property (ii) does not hold. Then, for each l in $\mathbb{N}$, a computation path κ_l exists, such that $|\kappa_l| \geq l$, and $\#[e, \kappa_l] \leq 1$ for each transition $e = (q_0, c, 0, q', r)$, where c is in Σ, q' is in K, and r is in $\{0, 1\}$. However, by the Pigeonhole principle, a transition f in D exists, such that

$$\#[f, \kappa_l] \geq \frac{l}{|D|}.$$

Thus, we may conclude that for all k in $\mathbb{N}$, a computation path κ_{l_k} exists, such that

$$\left|\#[e, \kappa_{l_k}] - \#[f, \kappa_{l_k}]\right| \geq k.$$

This clearly contradicts the assumption that the automaton A is strictly transition-$\mathcal{C}$-equiloaded.

For the cases $\mathcal{S} = \mathcal{A}$ and $\mathcal{S} = \mathcal{E}$, an analogous reasoning can be used to obtain a contradiction, if we suppose that the property (ii') resp. (ii'') does not hold.

Now, let us prove the easier right-to-left implications. We consider to be obvious that if the properties (i) and (ii) are satisfied, then the inequality

$$|\#[e, \gamma] - \#[f, \gamma]| \leq M \tag{3.4}$$

holds for all computation paths γ, and each two transitions e, f in D. Similarly, if (i) and (ii') are satisfied, the inequality (3.4) holds for all accepting computation paths γ, and if (i) and (ii'') are satisfied, the inequality holds for all computation paths γ accepting by empty memory. The theorem is proved. □

In what follows, we shall show that the previous theorem is a good characterization of strictly transition-$\mathcal{S}$-equiloaded DOCA. That is, we shall prove that the characterization from the previous theorem is decidable: we shall present an algorithm deciding the strict transition-$\mathcal{S}$-equiloadedness of a given DOCA by deciding, if the characterization from Theorem 3.3.1 is satisfied.

The algorithm shall proceed as follows: first, it examines all computation paths γ of length at most

$$M = \max\{|D|, |K|(|K| + 1)\} + 1,$$

such that γ visits a configuration with the state q_0 and the counter value 0 only at its beginning. This is done by generating all possible computation paths. If a computation path that visits a configuration with the state q_0 and the counter value 0 for the second time is discovered, it is thrown away. However, before this is done, the algorithm checks if each transition is used exactly once in the computation path. If not, the property (i) is violated, and the automaton is not strictly transition-$\mathcal{S}$-equiloaded.

At the end of this part of the algorithm, all computation paths γ of length exactly M are generated, such that γ visits a configuration with the state q_0 and the counter value 0 only at

its beginning. It is obvious that the property (i) of the characterization from Theorem 3.3.1 is satisfied, if and only if none of these computation paths can be prolonged to a computation path ending in a configuration with the state q_0 and the counter value 0. Corollary 3.2.4, together with a simple fact that the counter value at the end of a computation path is always less than or equal to the length of the computation path, implies that to decide if this property holds, it suffices to check all possible prolongments by at most

$$(M+1) \cdot |K| + |K|^4 - 1$$

transitions. Moreover, it follows from Lemma 3.2.1 that the property (ii) of the characterization from Theorem 3.3.1 is satisfied, if and only if the set of computation paths generated by the first part of the algorithm is empty. Further, by Lemma 3.2.1, the property (ii') of the characterization is satisfied, if and only if none of these computation paths is accepting, and none of them can be prolonged to an accepting computation path. It follows from Corollary 3.2.4 that it suffices to check all possible prolongments by at most

$$(M+1) \cdot |K| + 2 \cdot |K|^4 - 1$$

transitions. Finally, again by Lemma 3.2.1, the property (ii'') of the characterization is satisfied, if and only if none of these computation paths is accepting by empty memory, and none of them can be prolonged to a computation path accepting by empty memory. Corollary 3.2.4 implies that it suffices to check all possible prolongments by at most

$$(M+1) \cdot |K| + |K|^4 - 1$$

transitions. These observations result in Algorithm 1.

Algorithm 1 Deciding a strict transition-$\mathcal{S}$-equiloadedness of a DOCA by checking if the characterization from Theorem 3.3.1 is satisfied.

Input: Finitely described DOCA $A = (K, \Sigma, \delta, q_0, F)$, name of the function $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$.
Output: TRUE, if the automaton A is strictly transition-$\mathcal{S}$-equiloaded, FALSE otherwise.

$M \leftarrow \max\{|D|, |K|(|K|+1)\} + 1$
$C \leftarrow \{$the empty computation path$\}$
for $i \leftarrow 1, M$ **do**
    $C' \leftarrow \emptyset$
    for each γ in C **do**
        Let S be the set of all computation paths γ' of length i, such that γ' is a computation path γ prolonged by one extra transition.
        $C' \leftarrow C' \cup S$
    end for
    $C \leftarrow C'$
    for each γ in C **do**
        if γ begins and ends in a configuration with the state q_0 and the counter value 0 **then**
            $C \leftarrow C - \{\gamma\}$
            if $\#[e, \gamma] \neq 1$ for some e in D **then**
                return FALSE
            end if
        end if
    end for
end for

Algorithm 1 continued.

```
19: if S = C then
20:     if C = ∅ then
21:         return TRUE
22:     else
23:         return FALSE
24:     end if
25: else if S = A then
26:     for i ← 1, (M + 1) · |K| + 2 · |K|^4 do
27:         for each γ in C do
28:             if γ ends in an accepting state or in q_0 with the counter value 0 then
29:                 return FALSE
30:             end if
31:         end for
32:         C' ← ∅
33:         for each γ in C do
34:             Let S be the set of all computation paths γ' of length i, such that γ' is a computation path γ prolonged by one extra transition.
35:             C' ← C' ∪ S
36:         end for
37:         C ← C'
38:     end for
39:     return TRUE
40: else if S = E then
41:     for i ← 1, (M + 1) · |K| + |K|^4 do
42:         for each γ in C do
43:             if γ ends in a configuration with the counter value 0 then
44:                 return FALSE
45:             end if
46:         end for
47:         C' ← ∅
48:         for each γ in C do
49:             Let S be the set of all computation paths γ' of length i, such that γ' is a computation path γ prolonged by one extra transition.
50:             C' ← C' ∪ S
51:         end for
52:         C ← C'
53:     end for
54:     return TRUE
55: end if
```

3.4 Further Lemmas

In this section, we shall state and prove some additional lemmas that we shall use mainly in our study of relations between various families of strictly $\mathcal{S}$-equiloaded DOCA-languages.

Lemma 3.4.1 Let $A = (K, \Sigma, \delta, q_0, F)$ be a strictly transition-$\mathcal{S}$-equiloaded DOCA, for some $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$, accepting an infinite language. Then, for each q in K, $\delta(q, c, 0)$ is defined for at most one c in $\Sigma \cup \{\varepsilon\}$.

Proof. By contradiction. Let us suppose that two distinct (and thus necessarily non-ε) symbols c, d in Σ exist, such that both $\delta(q, c, 0)$ and $\delta(q, d, 0)$ are defined for some state q in K. That

is, $(q, c, 0, \mathrm{pr}_1(\delta(q, c, 0)), \mathrm{pr}_2(\delta(q, c, 0)))$ and $(q, d, 0, \mathrm{pr}_1(\delta(q, d, 0)), \mathrm{pr}_2(\delta(q, d, 0)))$ are two distinct transitions in D.

Since the language $L(A)$ is infinite, it follows from the characterization given in Theorem 3.3.1 that

$$(q_0, w, 0) \vdash^+ (q_0, \varepsilon, 0),$$

for some word w in Σ^*. Moreover, the property (i) of the characterization from Theorem 3.3.1 has to hold. That is, for every computation path γ, such that $|\gamma| \geq 1$ and γ reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise, the property

$$\#[e, \gamma] = 1$$

holds for every transition e in D. Now, if $q = q_0$, this is a clear contradiction. Let us therefore suppose that $q \neq q_0$. Then, since there are at least two distinct transitions leading from q at the counter value 0, every such computation path γ has to visit the state q at least twice with the counter value 0. Now, let us consider γ to be fixed. Without loss of generality, let us suppose that the transition $(q, c, 0, \mathrm{pr}_1(\delta(q, c, 0)), \mathrm{pr}_2(\delta(q, c, 0)))$ is used before the transition $(q, d, 0, \mathrm{pr}_1(\delta(q, d, 0)), \mathrm{pr}_2(\delta(q, d, 0)))$ in the computation path γ. Then, obviously, there is a computation subpath γ' of γ, corresponding to some computation

$$(q, cu, 0) \vdash^+ (q, \varepsilon, 0)$$

for some u in Σ^*, such that a configuration with the state q_0 and with the counter value 0 is not visited by γ'. Clearly, γ' can be iterated arbitrarily many times as a subpath of γ. Thus, a computation path violating the property (i) exists, and that is a contradiction. □

Lemma 3.4.2 Let $A = (K, \Sigma, \delta, q_0, F)$ be a strictly transition-$\mathcal{S}$-equiloaded DOCA, for some $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$, accepting an infinite language. Then, a unique positive integer k in $\mathbb{N}^+$, a unique sequence of states $\{q_i\}_{i=1}^{k+1}$ in K^{k+1}, and a unique partition of the set of transitions $\{S_i\}_{i=1}^{k}$ in $\left(2^D\right)^k$ exists, such that the following statement holds: let γ be an arbitrary computation path of the automaton A, such that $|\gamma| \geq 1$ and γ reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise. Then, the sequence of states of the automaton A visited by γ with the counter value being 0 is identical to $\{q_i\}_{i=1}^{k+1}$, and the set of transitions used by γ between the i-th and $(i+1)$-th visit of a configuration with the counter value 0 is S_i, for $i = 1, \ldots, k$. Clearly, $q_1 = q_{k+1} = q_0$.

Proof. We shall first prove that k is unique. This can be easily observed since, according to Lemma 3.4.1, every computation path γ satisfying the imposed conditions has to visit each state q in K, such that $\delta(q, c, 0)$ is defined for some c in $\Sigma \cup \{\varepsilon\}$, exactly once (except $q = q_0$ that is visited twice). Clearly, no other state can be visited by γ with the counter value being 0, since the computation would get stuck.

Now, we shall prove that $\{q_i\}_{i=1}^{k+1}$ is unique. For the purpose of contradiction, let us suppose that $\{p_i\}_{i=1}^{k+1}$ is a sequence of states, such that it is followed (in the sense of the statement of the lemma) by some computation path γ_1, and $\{r_i\}_{i=1}^{k+1}$ is a different sequence of states, such that it is followed by some different computation path γ_2, where both computation paths γ_1, γ_2 satisfy the conditions imposed in the statement of the lemma. By what we have stated above, the second sequence can be viewed as a nonidentical permutation of the first sequence (however, this permutation is required to preserve the first and the last element). Thus, integers i, j in $\{1, \ldots, k\}$ exist, such that $i < j$, and $p_i = r_j$. Clearly, the computation path γ_1 can be cut after reaching a configuration with the state p_i and the counter value 0, and then continued as the computation path γ_2. However, the resulting computation path γ clearly contradicts the property that k is always unique, since the number of visits of a configuration with the counter value 0 by the computation path γ is strictly less than $k + 1$.

Finally, we shall prove that $\{S_i\}_{i=1}^{k}$ is unique. Since the automaton A is strictly transition-$\mathcal{S}$-equiloaded for some $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$, and since the accepted language $L(A)$ is infinite, it follows

from the characterization given in Theorem 3.3.1 that a computation path γ exists, such that $|\gamma| \geq 1$ and γ reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise. By the property (i) of the characterization from Theorem 3.3.1, the property $\#[e, \gamma] = 1$ has to hold for all e in D.

Let us suppose that $\{T_i\}_{i=1}^{k}$ is a sequence of sets of transitions, followed by (in the sense of the statement of the lemma) the computation path γ. By our assumption, at least one another sequence of sets of transitions $\{U_i\}_{i=1}^{k}$ exists, such that it is followed by some another computation path, say γ', such that it is nonempty, and such that it reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise. Then, at least one index j in $\{1, \ldots, k\}$ exists, such that $T_j \neq U_j$. Moreover, since the sequence $\{q_i\}_{i=1}^{k+1}$ is unique, it follows that the computation of the form $(q_j, u, 0) \vdash^* (q_{j+1}, \varepsilon, 0)$ exists for some u in Σ^*, such that it uses exactly all transitions from T_j (as in the computation path γ), and another computation of the form $(q_j, v, 0) \vdash^* (q_{j+1}, \varepsilon, 0)$ exists for some v in Σ^*, such that it uses exactly all transitions from U_j (as in the computation path γ').

Now, let us consider a computation path γ'' such that it is identical to γ except that between the states q_j and q_{j+1} it follows the path of γ', i.e., the corresponding computation is the above mentioned computation of the form $(q_j, v, 0) \vdash^* (q_{j+1}, \varepsilon, 0)$. Clearly, $|\gamma''| \geq 1$ and γ'' reaches a configuration with the state q_0 and the counter value 0 at its beginning, at its end, but not otherwise. However, it is obvious that the property $\#[e, \gamma] = 1$ cannot hold for all e in D. This contradicts the characterization of strictly transition-equiloaded DOCA given in Theorem 3.3.1. The lemma is proved. □

3.5 Families of Strictly $\mathcal{S}$-Equiloaded Languages

In this section, we shall examine the relations that hold between various families of strictly $\mathcal{S}$-equiloaded DOCA-languages and the relations of these families to some families of languages studied earlier in this thesis.

Theorem 3.5.1 The following strict inclusions hold:

1. $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C})$, 2. $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E}) \subsetneq \mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A})$.

Proof. We shall prove only the first statement, the proof of the second statement is analogous. Let L in $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$ be a strictly transition-$\mathcal{C}$-equiloaded DOCA-language accepted by empty memory. Let $A = (K, \Sigma, \delta, q_0, F)$ be a strictly transition-$\mathcal{C}$-equiloaded deterministic one-counter automaton, such that $N(A) = L$. We shall construct a strictly transition-$\mathcal{C}$-equiloaded deterministic one-counter automaton $A' = (K', \Sigma', \delta', q_0', F')$, such that $L(A') = L$, as follows: $K' = K \times \{old, new\}$, $\Sigma' = \Sigma$, the transition function δ' is defined for all q, q' in K, c in $\Sigma' \cup \{\varepsilon\}$ and r in $\{-1, 0, 1\}$ by

$$\begin{aligned}
\delta'((q, old), c, 1) &= ((q', old), r) \iff \delta(q, c, 1) = (q', r), \\
\delta'((q, old), \varepsilon, 0) &= ((q, new), 0), \\
\delta'((q, new), c, 0) &= ((q', old), r) \iff \delta(q, c, 0) = (q', r),
\end{aligned}$$

$q_0' = (q_0, old)$, and $F' = K \times \{new\}$. The idea behind this construction is to replace each state of the automaton A by two states: the „old" one and the „new" one. If the counter value is greater than zero, then the computation proceeds by using old states, exactly as in the automaton A. However, when the counter value 0 is reached, the computation first makes an ε-transition from the old state to the new states and only then continues to simulate the computation of the automaton A (by using old states, again). The reason why the new states are introduced is that a state (q, new) of the automaton A' can be reached after reading some w in Σ^* if and only if some configuration of the automaton A with the state q and the counter value 0 can be reached after

reading w. Thus, if A accepts w by empty memory, then some new state of the automaton A' can be reached after reading w. Conversely, if some new state of A' can be reached after reading w, then A accepts w by empty memory. It therefore suffices to define the set F' to be $K \times \{new\}$.

Conversely, every language in $\mathcal{N}_{\delta-SEQ-DOCA}(\mathcal{C})$ has to contain the empty word ε. However, it is easy to find a language L in $\mathcal{L}_{\delta-SEQ-DOCA}(\mathcal{C})$, such that ε is not in L. The language $L = \{a\}^+$ may serve as an easy counterexample. The construction of a strictly transition-$\mathcal{C}$-equiloaded DOCA accepting L is easy and left to the reader. The theorem is proved. □

Theorem 3.5.2 The following strict inclusions hold:

1. $\mathcal{L}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathcal{L}_{\delta-SEQ-DOCA}(\mathcal{A})$, 2. $\mathcal{N}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathcal{N}_{\delta-SEQ-DOCA}(\mathcal{E})$.

Proof. The improper inclusions follow immediately from Theorem 1.6.3. We shall prove that these inclusions are proper.

Let us consider the language $L = \{abcd, acbd, bacd, bcad\}^*$. We shall prove that this language is in $\mathcal{L}_{\delta-SEQ-DOCA}(\mathcal{A})$, but not in $\mathcal{L}_{\delta-SEQ-DOCA}(\mathcal{C})$.

We shall construct a deterministic one-counter automaton $A = (K, \Sigma, \delta, q_0, F)$, such that A is strictly transition-$\mathcal{A}$-equiloaded, and such that $L(A) = L$ as follows: the set of states K shall be defined by

$$K = \{q_i \mid i = 0, \ldots, 21\} \cup \{p_{1,i} \mid i = 1, \ldots, 4\} \cup \{p_{2,i} \mid i = 1, \ldots, 5\} \cup \{p_{3,i} \mid i = 1, \ldots, 9\},$$

the alphabet Σ shall be $\Sigma = \{a, b, c, d\}$, the transition function δ shall be defined as follows:

$$\begin{array}{lll}
\delta(q_0, \varepsilon, 0) = (q_1, +1), & \delta(q_1, a, 1) = (p_{1,1}, +1), & \\
\delta(q_1, b, 1) = (p_{2,1}, +1), & \delta(q_1, c, 1) = (p_{3,1}, -1), & \\
\delta(q_1, d, 1) = (q_2, -1), & \delta(q_i, \varepsilon, 1) = (q_{i+1}, -1), & i = 2, \ldots, 20, \\
\delta(q_{21}, \varepsilon, 0) = (q_0, 0), & \delta(p_{1,i}, \varepsilon, 1) = (p_{1,i+1}, +1), & i = 1, \ldots, 3, \\
\delta(p_{1,4}, \varepsilon, 1) = (q_1, +1), & \delta(p_{2,i}, \varepsilon, 1) = (p_{2,i+1}, +1), & i = 1, \ldots, 4, \\
\delta(p_{2,5}, \varepsilon, 1) = (q_1, +1), & \delta(p_{3,i}, \varepsilon, 1) = (p_{3,i+1}, +1), & i = 1, \ldots, 8, \\
\delta(p_{3,9}, \varepsilon, 1) = (q_1, +1), & &
\end{array}$$

and the set F of accepting states shall consist only of the initial state, i.e., $F = \{q_0\}$. It is not hard to see that this automaton indeed accepts the language L: to arrive from the state q_1 to the only accepting state q_0, it is first obviously necessary to reach a configuration with the state q_1 and with the counter value 20. When the state q_1 is reached for the first time after the last visit of the state q_0, the counter value is always 1. That is, before leaving the state q_1 for the last time before the next visit of q_0, the counter value has to be increased by the value of 19.

There are three "cycles" from the state q_1: one consists of the states $p_{1,i}, i = 1, \ldots, 4$, and the character a is read during the cycle, second of the states $p_{2,i}, i = 1, \ldots, 5$, and the character b is read during the cycle, and third of the states $p_{3,i}, i = 1, \ldots, 9$, and the character c is read. The counter value is increased by the value of 5 in the first cycle, by the value of 6 in the second cycle, and by the value of 8 in the third cycle. The reader may easily convince himself that the only way how to express the number 19 as an integer conical combination of numbers 5, 6 and 8 is

$$5 + 6 + 8 = 19.$$

Thus, to increase the counter value by the value of 19, the computation has to go through each of these cycles exactly once. However, the third cycle cannot be used as the first of these cycles, since it can be easily seen that if the initial counter value is 1, the computation gets stuck in this cycle. After going through these cycles, the character d is always read. Thus, the statement $L(A) = L$ is proved. Moreover, the above reasoning also clearly implies that the automaton A is strictly transition-$\mathcal{A}$-equiloaded. That is, L is in $\mathcal{L}_{\delta-SEQ-DOCA}(\mathcal{A})$.

Now we shall prove that the language L is not in $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C})$. For the purpose of contradiction, let us suppose that a strictly transition-$\mathcal{C}$-equiloaded deterministic one-counter automaton $A' = (K', \Sigma, \delta', q'_0, F')$ exists, such that $L(A') = L$ (we can clearly assume that the alphabet of A' is Σ, since a transition on a character not in Σ would either spoil the strict transition-$\mathcal{C}$-equiloadedness, or change the accepted language).

The requirement of determinism of the automaton A' implies that every computation of the automaton A' begins with a sequence of ε-transitions and then arrives to some configuration with a state q and a counter value t, where it can choose (at least) between some a-transition and some b-transition (but there is not any ε-transition). Moreover, Lemma 3.4.1 implies that $t \geq 1$.

That is, the setting is as follows: for some q in K' and $t \geq 1$, we have $(q_0, \varepsilon, 0) \vdash^+ (q, \varepsilon, t)$. Moreover, for some q_a, q_b in K' and r_a, r_b in $\{-1, 0, +1\}$, $e_a = (q, a, 1, q_a, r_a)$ and $e_b = (q, b, 1, q_b, r_b)$ are transitions of A'.

Since the automaton A' is strictly transition-$\mathcal{C}$-equiloaded, the properties (i) and (ii) of the characterization given in Theorem 3.3.1 have to hold. Since, in addition, an arbitrarily long word beginning by a (resp. b) can be found in $L(A')$, this implies that if the transition e_a is chosen, the computation path has to be able to return to the transition e_b, and vice versa. That is, words w_a and w_b in Σ^* exist, such that

$$(q, aw_a, t) \vdash (q_a, w_a, t + r_a) \vdash^* (q, \varepsilon, t_a) \tag{3.5}$$

and

$$(q, bw_b, t) \vdash (q_b, w_b, t + r_b) \vdash^* (q, \varepsilon, t_b) \tag{3.6}$$

for some t_a, t_b in $\mathbb{N}$. Clearly, the transitions e_a and e_b can be used in an arbitrary order. Thus, it follows from Lemma 3.4.2 that the counter value is always positive in both the computation (3.5) and the computation (3.6). Furthermore, the inequalities $0 < t_a < t$ and $0 < t_b < t$ have to hold, since otherwise either the counter value 0 would be reached, or the "cycles" (3.5) and (3.6) could be "executed" arbitrarily many times and the automaton A' would not be strictly transition-$\mathcal{C}$-equiloaded (the property (ii) of the characterization from Theorem 3.3.1 would be violated).

Now, let us examine the possible lengths of the word w_a. The length of the word w_a clearly cannot be $4k + 3$ for some k in $\mathbb{N}$, since otherwise the computation clearly could continue by the use of some a-transition. However, the only a-transition leading from q at the counter value greater than 0 is e_a. But if this transition was used, the property (i) of the characterization from Theorem 3.3.1 would be violated.

If the length of the word w_a is $4k + 2$ for some k in $\mathbb{N}$, then the "accepting branch" of the computation would have to continue by some d-transition leading from q at the counter value greater than 0. However, the transition e_b would still be unused. This implies that this computation as well as the word w_a can be prolonged so that the prolonged length of w_a is $4k + 1$ or $4k$ for some k in $\mathbb{N}$.

If the length of the word w_a is $4k$ for some k in $\mathbb{N}$, then, clearly, the computation can continue by the use of some c-transition. That is, there is a transition $e_c = (q, c, 1, q_c, r_c)$, for some q_c in K' and r_c in $\{-1, 0, 1\}$. For the same reasons as above, a word w_c in Σ^* has to exist, such that

$$(q, cw_c, t_a) \vdash (q_c, w_c, t_a + r_c) \vdash^* (q, \varepsilon, t_c), \tag{3.7}$$

and such that the counter value is always positive during this computation. However, since $t > t_a$, it is also possible to reverse the order of the cycles, that is, to first use the c-cycle and only then to use the a-cycle. Instead of the word aw_acw_c, the word cw_caw_a would be read, but the terminal configuration would be exactly the same. But since the word aw_acw_c can be prolonged[2] to some word from $L(A')$, the same is true for cw_caw_a. But this is a contradiction, since the first character of this word is c and there is not any such word in $L(A')$.

[2]This obviously holds for at least one possible w_a, since otherwise the contradiction with $L(A') = L$ could be easily reached. Without loss of generality, let us suppose that we have chosen such w_a.

Thus, the only possible length of the word w_a that is left, is $4k+1$ for some k in $\mathbb{N}$. However, it is clearly possible to construct an arbitrarily long word w in $L(A')$, such that for every k in $\mathbb{N}$, the character $w[4k+2]$, if defined, is c and not b. That is, the number of steps needed until the transition e_b is used in some accepting computation, is not bounded by any constant. This clearly contradicts the characterization given in Theorem 3.3.1.

Now, let us consider the language $L' = \{abcd, acbd, bacd, bcad\}^*\{\varepsilon, c\}$. Clearly, $N(A) = L'$. Furthermore, the automaton A is obviously strictly transition-$\mathcal{E}$-equiloaded. Thus, L' is in $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E})$.

To prove that L' is not in $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$, exactly the same reasoning as above can be used. The only minor difference is in the case when the word w_a is supposed to have the length $4k$ for some k in $\mathbb{N}$. Then, the contradiction is reached not only by noting that a word beginning by c would have been in $L(A')$, but by noting that a word *of length at least 2* beginning by c would have been in $L(A')$ (that is clearly not the case for the language L'). The theorem is proved. □

In what follows, we shall examine the relations of the families of strictly transition-$\mathcal{S}$-equiloaded DOCA-languages to some families of languages studied in the previous chapter. We shall observe that the computational power of (state-accepting) strictly transition-$\mathcal{S}$-equiloaded DOCA is greater than the computational power of strictly transition-$\mathcal{S}$-equiloaded DFA or DFAε. However, on the other hand, we shall also make one slightly less optimistic observation: strictly transition-$\mathcal{S}$-equiloaded DOCA accept only some proper subset of the family of regular languages.

Theorem 3.5.3 The following strict inclusions hold:

1. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C}) \supsetneq \mathscr{L}_{\delta-SEQ-DFA\varepsilon}$,
2. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A}) \supsetneq \mathscr{L}_{\delta-SEQ-DFA\varepsilon}$.

Proof. It clearly suffices to prove the first statement: the second statement is a clear corollary of the first statement and of Theorem 3.5.2.

First, let us prove the improper inclusion. Let L in $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ be a strictly transition-equiloaded DFAε-language. Recall that $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ is defined by

$$\mathscr{L}_{\delta-SEQ-DFA\varepsilon} := \mathscr{L}_{\delta-SEQ-DFA\varepsilon}(\mathcal{C}) = \mathscr{L}_{\delta-SEQ-DFA\varepsilon}(\mathcal{A}).$$

Thus, there exists a strictly transition-$\mathcal{C}$-equiloaded deterministic finite automaton with ε-transitions $A = (K, \Sigma, \delta, q_0, F)$, such that $L(A) = L$. Moreover, we can clearly suppose that the graphical representation of the automaton A is connected. Thus, by Theorem 2.2.2, the graphical representation of the automaton A either does not contain any reachable directed cycle, or is a directed cycle through all states.

Let us define a strictly transition-$\mathcal{C}$-equiloaded deterministic one-counter automaton $A' = (K', \Sigma', \delta', q_0', F')$ accepting L as follows: $K' = K$, $\Sigma' = \Sigma$, $q_0' = q_0$, $F' = F$, and

$$\forall p, q \in K' \; \forall c \in \Sigma' \cup \{\varepsilon\} : \delta'(p, c, 0) = (q, 0) \iff \delta(p, c) = q$$

(transitions for the counter values greater than zero are left undefined). That is, the DOCA A' merely simulates the DFAε A with the counter value being constantly 0. The statement $L(A') = L$ is obvious.

It remains to prove that the deterministic one-counter automaton A' is strictly transition-$\mathcal{C}$-equiloaded. As we have already noted, the graphical representation of the automaton A either does not contain any reachable directed cycle, or is a directed cycle through all states.

If the graphical representation of the automaton A does not contain any reachable directed cycle, then, clearly, there is not any word w in Σ^*, such that $(q_0, w) \vdash_A^* (q_0, \varepsilon)$. By the definition of the automaton A', this implies that there is not any word w in $(\Sigma')^* = \Sigma^*$, such that $(q_0, w, 0) \vdash_{A'}^* (q_0, \varepsilon, 0)$. Thus, the property (i) of the characterization given in Theorem 3.3.1 is trivially satisfied. Moreover, there clearly is a nonnegative integer k in $\mathbb{N}$, such that for every computation path γ of the automaton A, the inequality $|\gamma| \leq k$ holds. Thus, the same property holds also for every

computation path of the automaton A'. In other words, the property (ii) of the characterization given in Theorem 3.3.1 is satisfied as well and the deterministic one-counter automaton A' is strictly transition-$\mathcal{C}$-equiloaded.

Similarly, if the graphical representation of the automaton A is a directed cycle through all states, it follows directly from the definition of the automaton A' that the properties (i) and (ii) of the characterization given in Theorem 3.3.1 are satisfied. That is, the deterministic one-counter automaton A' is again strictly transition-$\mathcal{C}$-equiloaded.

Let us prove that the inclusion is proper. In Example 3.1.2, we have observed that the language $L = \{ab, ba\}^*$ is in $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C})$. We shall prove that L is not in $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$.

For the purpose of contradiction, let us suppose that L is in $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$. Since the language L is infinite, a DFAε $A = (K, \Sigma, \delta, q_0, F)$ with the graphical representation of the form of a directed cycle through all states exists, such that $L(A) = L$. Let q in K be the first (in the direction of the directed cycle) state of the automaton A, such that at least one non-ε transition (q, c, q') in D leads from q. Such a state has to exist, since otherwise the accepted language $L(A)$ would be finite. If $c = a$, then every word in $L(A)$ begins by a, and that contradicts the assumption $L(A) = L$. An analogous contradiction can be reached also in the case $c = b$. Clearly, these are the only two possibilities. Thus, L is not in $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$. The theorem is proved. □

Corollary 3.5.4 The following strict inclusions hold:

1. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C}) \supsetneq \mathscr{L}_{\delta-SEQ-DFA}$,
2. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A}) \supsetneq \mathscr{L}_{\delta-SEQ-DFA}$.

Proof. Follows directly from Theorem 3.5.3 and Theorem 2.2.8. □

In order to make the length of this thesis reasonable, we shall omit proofs of the following theorems. The proofs may be found in our report [26].

Theorem 3.5.5 The following relations hold:

1. The families $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ are incomparable.
2. The families $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$ are incomparable.
3. The families $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{\delta-SEQ-DFA}$ are incomparable.
4. The families $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{L}_{\delta-SEQ-DFA}$ are incomparable.

Theorem 3.5.6 The following strict inclusions hold:

1. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathscr{R}$,
2. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A}) \subsetneq \mathscr{R}$,
3. $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathscr{R}$,
4. $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E}) \subsetneq \mathscr{R}$.

Now, let us turn our attention to the families of strictly state-$\mathcal{S}$-equiloaded DOCA-languages. We shall continue in omitting the proofs that may be found in our report [26].

Theorem 3.5.7 The following relations hold:

1. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$ are incomparable.
2. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$ are incomparable.
3. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$ are incomparable.
4. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$ are incomparable.

Theorem 3.5.8 The following strict inclusion holds:

$$\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathscr{N}_{K-SEQ-DOCA}(\mathcal{E}).$$

We leave the relation between the analogous families of languages accepted by accepting state open.

Open Problem 3.5.9 What is the relation between the family $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$ and the family $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$? Clearly, Theorem 1.6.3 implies $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C}) \subseteq \mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$. Is this inclusion proper or are these two families equal?

Theorem 3.5.10 The following strict inclusions hold:

1. $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C}) \supsetneq \mathscr{L}_{K-SEQ-DFA\varepsilon}$,
2. $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A}) \supsetneq \mathscr{L}_{K-SEQ-DFA\varepsilon}$.

Corollary 3.5.11 The following strict inclusions hold:

1. $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C}) \supsetneq \mathscr{L}_{K-SEQ-DFA}$,
2. $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A}) \supsetneq \mathscr{L}_{K-SEQ-DFA}$.

Theorem 3.5.12 The following relations hold:

1. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{K-SEQ-DFA\varepsilon}$ are incomparable.
2. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{L}_{K-SEQ-DFA\varepsilon}$ are incomparable.
3. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{K-SEQ-DFA}$ are incomparable.
4. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{L}_{K-SEQ-DFA}$ are incomparable.

Theorem 3.5.13 The following relations hold:

1. The families $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{R}$ are incomparable.
2. The families $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$ and $\mathscr{R}$ are incomparable.
3. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{R}$ are incomparable.
4. The families $\mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$ and $\mathscr{R}$ are incomparable.

In what follows, we shall focus on the relation between the families of strictly state-$\mathcal{S}$-equiloaded DOCA-languages and the families of strictly transition-$\mathcal{S}$-equiloaded DOCA-languages. In Theorem 2.2.9, we have observed that in the case of deterministic finite automata, strictly state-equiloaded automata have greater computational power than strictly transition-equiloaded automata. In the following theorem, we shall prove that this observation generalizes also to the case of deterministic one-counter automata.

Theorem 3.5.14 The following strict inclusions hold:

1. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$,
2. $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A}) \subsetneq \mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$,
3. $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C}) \subsetneq \mathscr{N}_{K-SEQ-DOCA}(\mathcal{C})$,
4. $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E}) \subsetneq \mathscr{N}_{K-SEQ-DOCA}(\mathcal{E})$.

Proof. We shall prove all four statements at once. First, let us prove the improper inclusions. Let $A = (K, \Sigma, \delta, q_0, F)$ be a strictly transition-$\mathcal{S}$-equiloaded deterministic one-counter automaton, for some $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$. We shall construct a strictly state-$\mathcal{S}$-equiloaded deterministic one-counter automaton $A' = (K', \Sigma', \delta', q_0', F')$, such that $L(A') = L(A)$, and $N(A') = N(A)$.

The idea behind the construction shall be as follows: the set of states of the automaton A' shall be precisely the set of transitions of the automaton A. Moreover, the following invariant shall be

hold: if a nonempty computation path γ of the automaton A on a word w ends by going through a transition e in D_A and with the counter value being t, then the corresponding computation path γ' of the automaton A' on the word w ends in the *state* e in $K' = D_A$ and with the couter value being t as well.

Two further details need to be addressed. The initial state of the automaton A' shall be an arbitrary transition of the automaton A leading to the state q_0 (we shall suppose that such a transition exists; otherwise, the accepted language would be necessarily finite and a discussion would become more-or-less trivial). Finally, the set of accepting states of the automaton A' shall be the set of all transitions of the automaton A leading to accepting states.

The formal construction is as follows: $K' = D_A$, $\Sigma' = \Sigma$, the transition function δ' is defined for p, q in K, c, d in Σ, t, t' in $\{0, 1\}$ and r' in $\{-1, 0, 1\}$ by

$$\delta'((p, c, t', q, r'), d, t) = ((q, d, t, \mathrm{pr}_1(\delta(q, d, t)), \mathrm{pr}_2(\delta(q, d, t))), \mathrm{pr}_2(\delta(q, d, t)))$$

if $\delta(q, d, t)$ is defined for the automaton A. If it is not defined, then the transition function δ' is left undefined for the corresponding inputs as well. Moreover, q'_0 is defined to be an arbitrary member of the set

$$\{(p, c, t, q, r) \in D_A \mid q = q_0\}.$$

Finally, the set of accepting states F' is defined by

$$F' = \{(p, c, t, q, r) \in D_A \mid q \in F\}.$$

It is obvious that the above mentioned invariant holds. Thus, it is not hard to see that indeed $L(A') = L(A)$ and $N(A') = N(A)$, and that the automaton A' is strictly state-$\mathcal{S}$-equiloaded (assuming that A is strictly transition-$\mathcal{S}$-equiloaded).

Now, let us prove that these inclusions are proper. The statement is a direct corollary of Theorem 3.5.6 and of Theorem 3.5.13. To be more specific, in Theorem 3.5.13 we have proved that each of the four families on the right side of our relations contain at least one nonregular language. However, in Theorem 3.5.6 we have proved that none of the four families of languages on the left side contain any nonregular language. The theorem is proved. □

3.6 Closure Properties

In this section, we shall summarize the closure properties of the families of strictly $\mathcal{S}$-equiloaded DOCA-languages. We shall omit proofs that can be found in our report [26]. The closure properties that are not stated in this section are open up to now.

Theorem 3.6.1 None of the families $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C})$, $\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A})$, $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$, and $\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E})$ is closed under concatenation, union, complementation, closure, positive closure, reversal, and inverse homomorphism.

Theorem 3.6.2 The families $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$ and $\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$ are not closed under concatenation, complementation, reversal, and inverse homomorphism.

Conclusion

In this thesis, we have presented and analyzed several definitions of balanced use of resources in deterministic sequential computations. The main aims of this thesis have been to generalize the earlier definitions for DFA, presented in [27], [28] and [25] to higher models of computation, to unify the earlier theories of state-equiloadedness and transition-equiloadedness for DFA, and to initiate the study of equiloadedness for some model of computation accepting nonregular sets. We may conclude that these aims have been successfully achieved.

In Section 1.2, we have defined a new abstract model of computation, the *abstract deterministic automata* (ADA). ADA are an AFA-inspired abstraction of deterministic automata with a one-way input tape and some kind of auxiliary memory. Many widely used models of computation, e.g., DFA, DFAε, DOCA, DPDA, or some variants of deterministic Turing machines, can be viewed as special cases of ADA. This has enabled us to present our definitions of equiloadedness independently from a particular model of computation – the definition for ADA applies to all models of computation that we examine in this thesis.

Next, in Section 1.4 and Section 1.5, we have presented our definitions of *strict $\mathcal{S}$-equiloadedness*, *$\mathcal{S}$-equiloadedness*, and *weak $\mathcal{S}$-equiloadedness*. All of these definitions have been presented for ADA and both for states and for transitions. These definitions generalize the earlier definitions from [25], [27] and [28]. This generalization is twofold: first, the definitions are stated for ADA instead of DFA. Second, in the earlier works, only accepting computation paths have been taken into account. Our definitions include the parameter $\mathcal{S}$ that makes it possible to specify the set of computation paths that we are interested in.

Further, we have examined some properties of equiloadedness that hold for abstract deterministic automata in general. Most importantly, we have proved some relations between the families of equiloaded languages that hold for every model of computation that is a special case of ADA. Moreover, we have defined the concept of prefix-dense languages that can be used to prove that a given language is not strictly $\mathcal{S}$-equiloaded for any model of computation that is a special case of ADA.

Later in this thesis, we have studied several families of equiloaded DFA, DFAε and DOCA, as well as the corresponding families of equiloaded languages. Equiloaded DFA have been studied already in the earlier works [27], [28] and [25], however we have presented some new results. Equiloaded DFAε and DOCA have not been studied yet.

For several families of equiloaded automata, we have proved their characterizations. Table Concl.1 contains the summary of these characterizations, including the numbers of corresponding theorems. Among characterizations, probably the most important results are the characterization of weakly state-$\mathcal{C}_{=}$-equiloaded DFA and DFAε (Theorem 2.3.31) and the characterization of strictly transition-$\mathcal{S}$-equiloaded DOCA for $\mathcal{S}$ in $\{\mathcal{C}, \mathcal{A}, \mathcal{E}\}$ (Theorem 3.3.1). Both these characterizations are completely new. The first of these characterization emphasizes the importance of Perron-Frobenius eigenvalues related to DFA and DFAε as a characteristic that can be used to analyze various quantitative properties of finite automata. The second is up to now the only known characterization of equiloaded DOCA. Since it characterizes the families of strictly transition-$\mathcal{S}$-equiloaded DOCA by some properties of computation paths, the decidability of this characterization have not been immediately clear. However, we have proved this decidability and presented an algorithm that decides if a given DOCA is strictly transition-$\mathcal{S}$-equiloaded by deciding if the conditions imposed in the characterization from Theorem 3.3.1 are satisfied.

The Model	Type of Equiloadedness	Resource	For $\mathcal{S}$ in	Characterization
DFA and DFAε	Strict $\mathcal{S}$-Equiloadedness	States	$\mathcal{C}, \mathcal{A}$	Theorem 2.2.1
		Transitions	$\mathcal{C}, \mathcal{A}$	Theorem 2.2.2
	Weak $\mathcal{S}$-Equiloadedness	States	$\mathcal{C}_=$	Theorem 2.3.31
			$\mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq$	?
		Transitions	$\mathcal{C}_=, \mathcal{A}_=$	Theorem 2.3.19
			$\mathcal{C}_\leq, \mathcal{A}_\leq$	Theorem 2.3.20
	$\mathcal{S}$-Equiloadedness	States	$\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq$	?
		Transitions	$\mathcal{C}_=, \mathcal{A}_=, \mathcal{C}_\leq, \mathcal{A}_\leq$	?
DOCA	Strict $\mathcal{S}$-Equiloadedness	States	$\mathcal{C}, \mathcal{A}, \mathcal{E}$	?
		Transitions	$\mathcal{C}, \mathcal{A}, \mathcal{E}$	Theorem 3.3.1

Table Concl.1: The summary of theorems providing characterizations of the families of equiloaded automata studied in this thesis.

In Section 2.1, we have observed that several basic quantities used in our study of $\mathcal{S}$-equiloaded DFA and DFAε may be computed as solutions to initial value problems for homogeneous systems of first-order linear OΔEs with constant coefficients. Moreover, we have observed some relations between the matrices of these systems and the transition matrix of a given automaton. Since systems of this kind can be solved relatively easily, this have lead us to the elegant mathematical method of computing closed forms of these basic quantities, and to the numerical algorithm of computing equiloadedness measures for DFA and DFAε. Furthermore, since the matrices of the presented systems are all nonnegative, the results obtained in Section 2.1 have enabled us to use the Perron-Frobenius theory to study the asymptotic properties of these basic quantities.

In Subsection 2.3.1, we have proved that the alternative definition of $\mathcal{S}$-equiloaded DFA and DFAε, based on the definitions given in [27] and [28], is equivalent to our definition of $\mathcal{S}$-equiloadedness. This result unifies the earlier theory of state-equiloaded DFA ([27] and [28]) with the earlier theory of transition-equiloaded DFA ([25]).

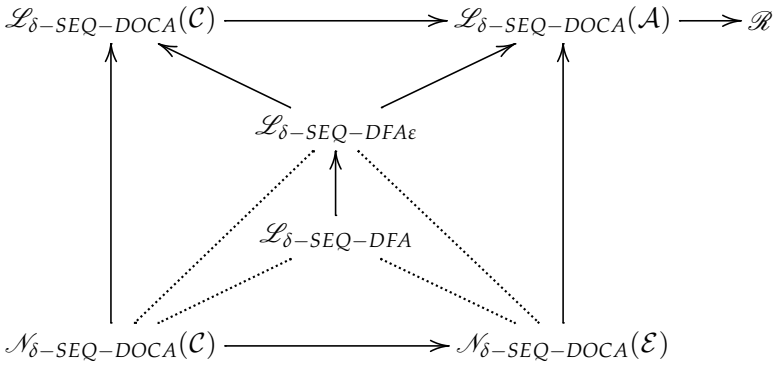


Figure Concl.1: The diagram of relations between various families of strictly transition-$\mathcal{S}$-equiloaded languages. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are families of languages, the arrow $\mathscr{L}_1 \to \mathscr{L}_2$ is supposed to be read as a proper inclusion $\mathscr{L}_1 \subsetneq \mathscr{L}_2$. A dotted line between two families of languages indicates that the families are incomparable.

Relations between various families of equiloaded languages have also been a subject of our study. We have proved numerous results on inclusions, strict inclusions, identities and incomparability relations between the families of equiloaded languages. We depict some of them in diagrams. In Figure Concl.1, the diagram representing the relations between the families of

strictly transition-$\mathcal{S}$-equiloaded languages is shown. The diagram in Figure Concl.2 represents the relations between the analogous families of strictly state-$\mathcal{S}$-equiloaded languages. Finally, the diagram in Figure Concl.3 shows the relations between the families of transition-equiloaded DFA-languages and DFAε-languages.

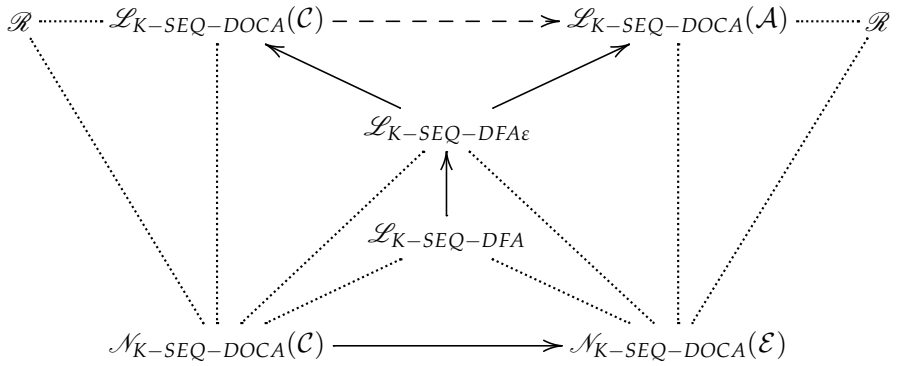


Figure Concl.2: The diagram of relations between various families of strictly state-$\mathcal{S}$-equiloaded languages. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are families of languages, an arrow $\mathscr{L}_1 \to \mathscr{L}_2$ is supposed to be read as a proper inclusion $\mathscr{L}_1 \subsetneq \mathscr{L}_2$. A dashed arrow $\mathscr{L}_1 \dashrightarrow \mathscr{L}_2$ is an inclusion $\mathscr{L}_1 \subseteq \mathscr{L}_2$ with a strict inclusion being open. A dotted line between two families of languages indicates that the families are incomparable.

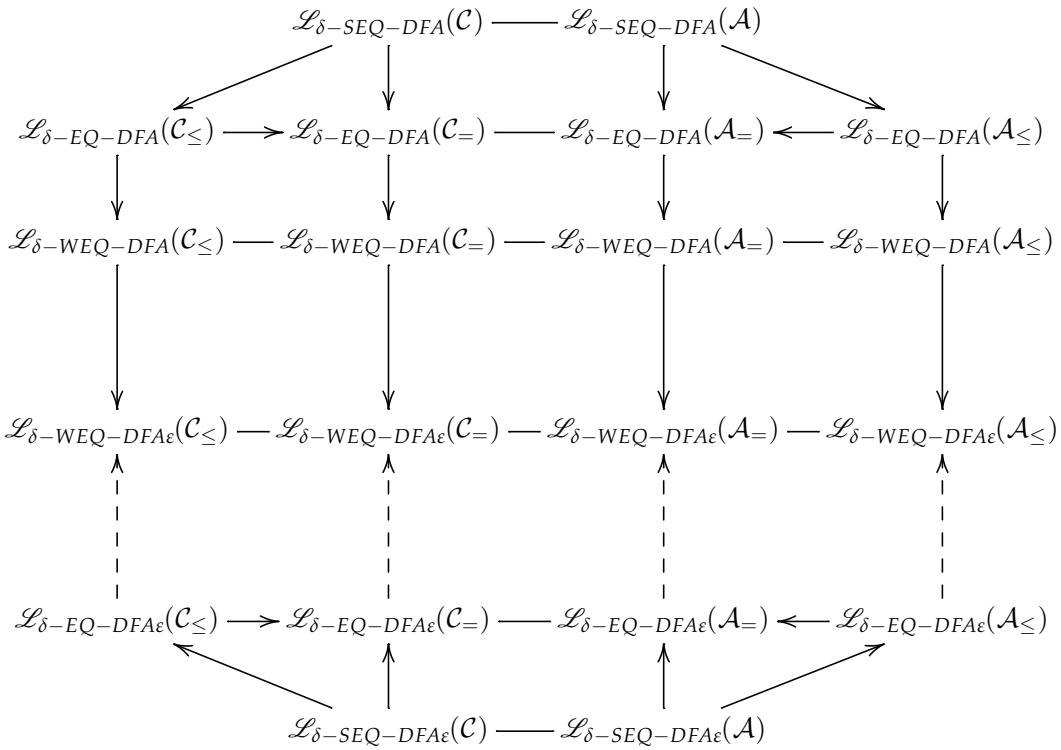


Figure Concl.3: The diagram of relations between various families of transition-equiloaded DFA(ε)-languages. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are families of languages, an arrow $\mathscr{L}_1 \to \mathscr{L}_2$ is supposed to be read as a proper inclusion $\mathscr{L}_1 \subsetneq \mathscr{L}_2$. A dashed arrow $\mathscr{L}_1 \dashrightarrow \mathscr{L}_2$ is an inclusion $\mathscr{L}_1 \subseteq \mathscr{L}_2$ with a strict inclusion being open. A solid line between two families of languages indicates that the families are equal.

We have also studied the closure properties of several families of equiloaded languages. We

summarize our results in Table Concl.2. The families of languages, for which we have not studied the closure properties yet, are omitted from the table. Some of the closure properties presented in this table have already been proved earlier – the closure properties of $\mathscr{L}_{K-SEQ-DFA}$ and $\mathscr{L}_{K-EQ-DFA}(\mathcal{A}_{=})$ are due to [27] and [28], the closure properties of the families $\mathscr{L}_{\delta-SEQ-DFA}$, $\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_{=})$ and $\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{=})$ are due to [25].

	$\cdot$	$\cup$	$\cap$	C	$*$	$+$	R	h	h^{-1}
$\mathscr{L}_{K-SEQ-DFA}$	No	No	Yes	No	No	No	No	No	No
$\mathscr{L}_{K-SEQ-DFA\varepsilon}$	No	No	Yes	No	No	No	No	No	No
$\mathscr{L}_{\delta-SEQ-DFA}$	No	No	Yes	No	No	No	No	No	No
$\mathscr{L}_{\delta-SEQ-DFA\varepsilon}$	No	No	Yes	No	No	No	No	Yes	No
$\mathscr{L}_{K-EQ-DFA}(\mathcal{A}_{=})$	No	No	No	No	?	?	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{C}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA}(\mathcal{A}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{C}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-EQ-DFA\varepsilon}(\mathcal{A}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{=})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{C}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA}(\mathcal{A}_{\leq})$	No	No	No	No	No	No	No	No	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{A}_{=})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{C}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{\delta-WEQ-DFA\varepsilon}(\mathcal{A}_{\leq})$	No	No	No	No	?	?	?	?	No
$\mathscr{L}_{K-SEQ-DOCA}(\mathcal{C})$	No	?	?	No	?	?	No	?	No
$\mathscr{L}_{K-SEQ-DOCA}(\mathcal{A})$	No	?	?	No	?	?	No	?	No
$\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{C})$	No	No	?	No	No	No	No	?	No
$\mathscr{L}_{\delta-SEQ-DOCA}(\mathcal{A})$	No	No	?	No	No	No	No	?	No
$\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{C})$	No	No	?	No	No	No	No	?	No
$\mathscr{N}_{\delta-SEQ-DOCA}(\mathcal{E})$	No	No	?	No	No	No	No	?	No

Table Concl.2: The summary of closure properties of various families of equiloaded languages that have been proved up to now.

In our future research, we plan to focus mainly on the families of $\mathcal{S}$-equiloaded and weakly $\mathcal{S}$-equiloaded DOCA and to extend the theory also to some models of computation higher than DOCA. However, also some interesting open problems on equiloaded DFA(ε) and strictly $\mathcal{S}$-equiloaded DOCA have arisen in this thesis that may be as well a subject of a fruitful future research.

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