MASARYK UNIVERSITY
FACULTY OF INFORMATICS

New challenges in planar emulators

MASTER'S THESIS

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Declaration

I hereby declare that this thesis is my original work, which I have elaborated on my own. All sources, resources and literature which I have used or drawn from in the elaboration of this thesis are properly quoted in the thesis by a complete reference to the appropriate source.

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Abstract

This work, which follows up on our Bachelor's research, deals with the arising challenges in the field of planar emulations of graphs. We show that the non-projective graphs with finite planar emulators must be planar expansions of internally 4-connected graphs from a specific finite set, or contain one of five minor minimal non-projective graphs as a minor. Consequently, this set of graphs is listed. The work includes a detailed description of the used tools and their optimization. We also consider the problem of the existence of finite planar emulators for the class of cubic graphs showing that there are only two non-projective cubic graphs that can be planar-emulable.

Keywords

Finite planar emulator, Fellows' conjecture, graph minor, projective plane, finite planar cover

Contents

Chapter 1

Introduction

A graph is a discrete mathematical structure that consists of a collection of nodes and the connections between them. A very common representation of graphs is a drawing — the nodes are situated as points on some surface, usually the plane, and the connections are drawn as curves connecting those points. For some graphs, we can accomplish a drawing where no two edges intersect. Such graphs are called planar.

The graphs with a planar drawing are interesting not only for their easily readable representation. It was shown that many problems can be solved efficiently on graphs with a planar drawing as opposed to graphs in general.

The concept of finite planar emulations of graphs was introduced by M. Fellows in the 1980's [12]. A finite planar emulator is a way how the structure of a non-planar graph G can be interpreted by planar means. It is a graph H that, as opposed to G, is planar, and there is a projection $\varphi : H \to G$ that fixes the local structure of G in H (see Chapter 3 for precise definition).

It is rather easy to decide if a given graph is planar (see Theorem 2.5). However, determining whether a non-planar graph has a finite planar emulator turns out to be in many respects nontrivial. In 1988, Fellows pronounced a conjecture that developed into the claim that no graph without embedding in the projective plane is emulable in planar.

Despite the long time that has passed, there has been quite little accomplished in this field. Fellows' conjecture surprisingly fell in 2008 by the discovery of two finite planar emulators for non-projective graphs. Many more emulators for non-projective graphs were discovered later.

Having virtually no useful characterization of planar-emulable graphs, within our Bachelor's research, we attempted to follow the approach to a problem of similar fashion (finite planar covers and Negami's conjecture) taken by Hliněný and Thomas. Initial success showed that this direction seems to be very promising, however there are many challenges arising. This work tackles some of those challenges and summarizes the results of our ongoing research.

This introduction is followed by Chapter 2, which provides the basic notion of graph theory used in this work. In Chapter 3, we introduce the problem and give an overview of the results accomplished in this field, which this thesis is elaborated on. Especially, in Section 3.1, we summarize the methodology of our research. Some interesting details regarding the implementation of our tools are described in Chapter 5. In Chapter 6, we provide the results of our search aiming to characterize the non-projective graphs with finite planar emulators which do not contain a minor isomorphic to a graph from the $K_7 - C_4$ family. In the subsequent Chapter 7, we restrict the problem to a specific class of graphs — the non-projective cubic graphs. Chapter 8 provides conclusions and suggestions for future work.

Chapter 2

Basic notion

A *graph* is a collection of nodes, called *vertices*, and the connections between them, called *edges*. Formally, a graph is an ordered pair (V, E) where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. If G is a graph, we denote the set of vertices and edges of G by $V(G)$ and $E(G)$ respectively. Two vertices connected by an edge are referred to as *adjacent* to each other. If e is an edge, the ends of e are said to be *incident* with e.

The set of edges $E(G)$ is a binary relation on $V(G)$. A graph is called *undirected* if this relation is symmetric. The reflexive pairs in $E(G)$, i.e. the edges with identical ends, are usually referred to as *loops*. If $E(G)$ is irreflexive and symmetric, the graph is called *simple*. The number of vertices of a graph G is the cardinality of the set $V(G)$, and the number of edges is the cardinality of $E(G)$. A graph is said to be *finite* if both its vertex set and its edge set are finite. All the graphs considered in this work are finite and simple. The edges are denoted by $e = \{u, v\}$, or shortly by $e = uv$, for an edge e with ends u, v.

2.1 Graphs and their structure

If v is a vertex of graph G, the vertices adjacent to v in G are called *neighbours* of v. The set of all neighbours of v in G is called a *neighbourhood* denoted by $N(v)$. The degree $d_G(v)$ of a vertex v in G is the number of neighbours of v. In particular, a vertex of degree 3 is called *cubic*. Graph G is *k-regular* if all the vertices of G have degree k. A *cubic graph* is a graph that is 3-regular. From the combinatorial point of view, it can be easily seen that

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|.$$

Two graphs $G = (V, E)$ and $H = (V', E')$ are *identical* if $V = V'$ and $E = E'$. Two graphs H and G are said to be *isomorphic* if there is a bijection

$\phi : G \to H$ such that every edge $(u, v) \in E(G)$ is mapped to an edge $(\phi(u), \phi(v)) \in E(H)$. The projection ϕ is called an *isomorphism* between G and H. An *automorphism* is an isomorphism $\phi : G \to G$ of a graph to itself.

A *homomorphism* from H onto G is a projection $\varphi : H \to G$ (not necessarily bijective) such that $\varphi(V(H)) = V(G)$ and for every egde (u, v) in $E(H)$, there is an edge $(\varphi(u), \varphi(v))$ in $E(G)$.

Let $G = (V, E)$ be a graph. Any graph $F = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$ is called a *subgraph* of G, written as $F \subseteq G$. If E' is maximal, i.e. it contains all the edges between the vertices of V' that occur in G, graph F is called an *induced subgraph*[1] of G denoted by $G \restriction V'$. It is also said that V' *induces* F in G. Trivially, every graph G contains (induced) subgraph G and the empty graph. Subgraph F of G with $F \neq G$ is called a *proper subgraph*. If $V(F) = V(G)$ for $F \subseteq G$, F is a *spanning subgraph* of G.

An *independent set* in a graph G is a subset $I \subseteq V(G)$ of vertices that induce a graph with no edges. A *clique* in a graph G is a subgraph of G that is isomorphic to a complete graph (see the end of this section for the definition).

A *walk* in a graph G is a finite sequence $v_0 e_1 v_1 e_2 v_2 \ldots v_k$ such that e_i is an edge with ends v_{i-1}, v_i for $1 \leq i \leq k$. The vertices v_0, v_k are *ends* of the walk, and all other vertices are *internal vertices*. The walk is said to *connect* v_0 and v_k. The length of the walk is the number k. Note that a walk allows that $v_i = v_j$ for some $0 \leq i \leq j \leq k$. If G is simple, a walk can be denoted simply by the sequence of vertices $v_0, v_1 \ldots v_k$. A walk is said to be *closed* if $v_0 = v_k$. In this respect, a *cycle in a graph* is a closed walk such that all vertices $v_1 \ldots v_k$ are pairwise distinct.

A graph G is said to be *connected* if every pair of its vertices is connected by a path. A component of G is every inclusion-wise maximal induced subgraph of G that is connected. A *vertex cut*, or simply a *cut*, in a graph G is a set $X \subseteq V(G)$ such that $G - X$ has more components than G. A cut of size k is referred to as k*−cut*. A graph is called k-connected if it is connected,

1. The literature often denotes a subgraph of G induced by vertex set $V' \subseteq V(G)$ by $G[V']$. Our notation is however kept consistent with [18, 20] as we use the these publications in a fundamental way.

has at least $k + 1$ vertices, and has no cut of size less than k.

A *separation* in a graph G is a pair of sets (A, B) such that $A \cup B = V(G)$ and there is no edge between $A - B$ and $B - A$. A separation (A, B) is *nontrivial* if both $A - B$ and $B - A$ are nonemtpy. The *order of separation* (A, B) is the cardinality of $A \cap B$. A separation of size k is also called a $k-$*separation*. A connected graph has a nontrivial $k-$separation if and only if it has a $k-$cut.

Theorem 2.1 (Menger's theorem)**.** *Let G be a graph and $C, D \subseteq V(G)$. The maximum number of vertex disjoint paths between vertices from C and D is the minimum order of a separation $(A.B)$ in G such that $C \subseteq A$ and $D \subseteq B$.*

Corollary 2.2. *Graph G on $k + 1$ vertices is $k-$connected if and only if there are k vertex disjoint paths (up to their ends) between any two vertices u, v in G.*

Similarly to vertex connectivity, it also makes sense to define edge connectivity. As this notion in not utilized in any way in this work, we just remark that it exists and refer to literature (e.g. [8]) for more details.

If u, v are two vertices of a graph G, then $G + uv$ refers to the graph obtained by *adding an edge* e connecting u to v in G. Note that all graphs considered in this work are simple without any parallel edges, thus this operation does not modify G in any way if the edge u, v already exists. The graph obtained from G by adding a set of edges E is denoted by $G + E$. We also use $G^{+e_1+\dots}_{+\dots+e_k}$ for long sequences of k edge additions $e_1, \dots, e_k$. The notion of *removing edges* is analogous.

If G_1, G_2 are two graphs, the *union* of G_1, G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. A graph H results from graph G by *subdividing an edge* $e = uv$ with vertices $v_1, \dots, v_k$ that are not in $V(G)$ if $H = (G - e) \cup P$ where $P = uv_1 \dots v_k v$ is a path. Graph H is referred to as a subdivision of graph G. If graph F is isomorphic to a subdivision of graph G, it is said to be *homeomorphic* to G.

If G is a graph and $e = \{u, v\}$ is an edge in G, the graph F resulting from G by *contracting edge* e is described as follows: $V(F) = V(G) \setminus \{u, v\} \cup \{w_e\}$ where w_e is a new vertex, and the set of edges $E(F) = E(G - u - v) \cup \{w_e x \mid ux \in E(G) \vee ex \in E(G)\}$.

Graph G is a minor of H, denoted by $G \leq H$, if G can be obtained from a subgraph of H by contracting edges. The notion of graph minors plays a fundamental role in this work. The following proposition is widely exploited by our tools as described in Chapter 5.

Proposition 2.3. *G is a minor of H if and only if there are disjoint connected subgraphs $F_v \subseteq H$, $v \in V(G)$ such that whenever xy is an edge of G, there is an edge between $V(F_x)$ and $V(F_y)$ in H.*

Let $\mathcal{P}$ be a graph property. If for every graph G with property $\mathcal{P}$ holds that every minor F of G also has $\mathcal{P}$, then $\mathcal{P}$ is said to be *closed* under taking minors. The following is a famous result by N. Robertson and P. Seymour.

Theorem 2.4 (N. Robertson, P. Seymour)**.** *Let $\mathcal{P}$ be a minor closed property. Then $\mathcal{P}$ has a finite set of minor minimal obstructions Λ such that no graph with a minor isomorphic to a graph from Λ has property $\mathcal{P}$. Moreover, $\mathcal{P}$ is testable in time $\mathcal{O}(n^3)$.*

Unfortunately, Theorem 2.4 is purely of existential nature and hence provides no clues for constructing a cubic algorithm for *P*. Set Λ mentioned in the theorem is sometimes also called the *sporadic family* for $\mathcal{P}$. The graphs from Λ, and in a broad sense also the graphs that contain a minor from Λ, are also referred to as *forbidden minors* or *minor obstructions* for $\mathcal{P}$. In this thesis, we extensively work with the sporadic family of graphs that embed in the projective plane (see Section 2.2.2). The goal of this entire work can be viewed as finding the sporadic family for the graphs with finite planar emulators (cf. Chapter 3).

There is a convention in naming graphs. A graph with $n+1$ vertices connected by n edges to a *path* of length n is commonly denoted by P_n. A *cycle* of length n is a path P_{n-1} with one additional edge connecting vertices 1 and n, denoted by C_n. A cycle of length k is sometimes called a *k-cycle* and in particular 3-cycle is called a *triangle*. The graph K_n denotes a *complete graph* with n vertices where every two vertices are adjacent to each other.

A graph G is called *bipartite* if the vertex set can be partitioned into two sets $V(G) = A \cup B$ such that both $G \restriction A$ and $G \restriction B$ have no edges. The sets A, B are called *parts* of G. A graph $K_{m,n}$ refers to a complete bipartite graph

G with pairwise disjoint parts V_1, V_2 where every pair of vertices $u \in V_1$, $v \in V_2$ is connected by an edge. Such a graph is called a *complete bipartite* graph. The notion of (complete) bipartite graphs can be easily generalized to (complete) p-partite graphs.

2.2 Topological properties of graphs

A *surface* is a Hausdorff topological space S such that every point of S has an open neighbourhood homeomorphic to $\mathbb{R}^2$. A *curve*, sometimes also called an *arc*, is the image of a continuous function $f : [0,1] \to S$. The curve is simple if f is injective. If $f(0) = u$ and $f(1) = v$, the arc $A = f([0,1])$ is said to *join* or to *connect* u and v.

A graph G is *embedded* in surface S if the vertices of G are distinct elements of S and every edge of G is a simple curve connecting vertices that are adjacent in G. Furthermore, it is required that no two edges intersect in a point distinct from a common vertex, and that no edge contains an isolated vertex (vertex of degree 0 in G). An *embedding* of G in S is an isomorphism of G with a graph G' embedded in S. In this case, G' is said to be a *representation* of G in S and we say that *G can be embedded* into S. Note that if a graph G has an embedding in S, so does every minor of G. Thus, the existence of embeddings is minor-closed.

Let G be a graph embedded in a surface S. A *face* of G is an arcwise connected component of $S - G$. The set of faces is denoted by $F(G)$. Each face is bounded by a closed walk called a *facial walk*. A face f of G is said to be *incident* with vertices and edges contained in the boundary of f.

2.2.1 Planar graphs

A large group of graphs of our interest are the planar graphs. According to the definitions above, graph G is *planar* if it has an embedding in the sphere S_0. It is however more common to treat planar graphs as embeddings in the (Euclidean) plane $\mathbb{R}^2$. Note that as the plane is homeomorphic to any open disc in the sphere, both approaches are equivalent. In this work, we approach planar graphs as $\mathbb{R}^2$ embeddings.

Theorem 2.5 (Kuratowski, 1930). *A graph is planar if and only if it does not contain a subgraph isomorphic to a subdivision of* K_5 *or* $K_{3,3}$.

Corollary 2.6. *A graph is planar if and only if it does not contain a minor isomorphic to* K_5 *or* $K_{3,3}$.

The following theorem is a generally known result of Euler, which provides a necessary, but not sufficient condition for a graph to be planar.

Theorem 2.7 (Euler's formula). *Let G be a planar graph, then* $|V(G)| - |E(G)| + |F(G)| = 2$.

Proposition 2.8 can be easily derived from Euler's formula.

Proposition 2.8. *Let G be a planar graph.*

(a) If $|V(G)| \geq 3$, *then* $|E(G)| \leq 3|V(G)| - 6$.

(b) If $|V(G)|$ *and there is no triangle in G, then* $|E(G)| \leq 2|V(G)| - 4$.

(c) G contains a vertex of degree at most 5.

A *plane graph* is a graph embedded in the plane. The unbounded face of G is called the *outer face* of G. A graph G that can be embedded in the plane with every vertex on the outer face of G is called *outer planar*.

Theorem 2.9. *A graph is outer planar if and only if it does not contain a subgraph isomorphic to a subdivision of* K_4 *or* $K_{2,3}$.

2.2.2 Projective plane

A *manifold* is a topological space that is locally Euclidean (i.e., around every point, there is a neighborhood that is topologically the same as the open unit ball in $\mathbb{R}^n$).

The *projective plane* is a topological manifold that can be described by connecting the sides of a rectangle in the orientations (Figure 2.1). It can be also seen as the sphere S_0 (or the Euclidean plane, see. 2.2.1) with a single *crosscap* — a self-intersection of a one-sided surface.

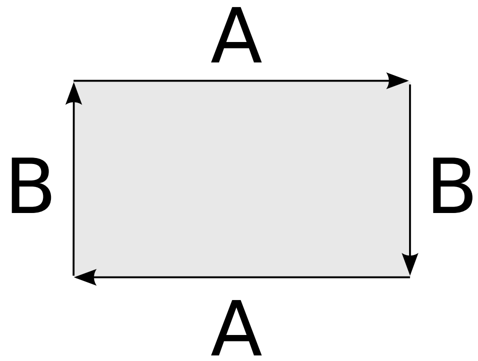

Figure 2.1: The projective plane can be obtained by unifying oriented edges of a rectangle.

A graph is called *projective planar* (or shortly *projective*) if it has an embedding in the projective plane. The class of projective graphs is strictly larger than the class of planar graphs. For example, both the graph K_5 and $K_{3,3}$, the minimal obstructions for planar graphs (cf. Theorem 2.5), can be embedded in the projective plane without any edge crossings.

Theorem 2.10 is an analog of Euler's formula (cf. Theorem 2.7) extended from the plane to the projective plane. In fact, this can be done for any surface S. The constant in $\chi(S) = |V(G)| - |E(G)| + |F(G)|$ is known as *Euler's characteristic* of surface S.

Theorem 2.10. *Let G be a graph with a projective embedding. Then, $|V(G)| - |E(G)| + |F(G)| = 1$. Consequently, G cannot contain more than $3|V(G)| - 3$ edges.*

Similarly to planar graphs, projective graphs can be characterized by obstructions. Glover, Huneke and Wang [17] found a family Λ of 35 graphs such that each member of Λ has no embedding in the projective plane and is minor minimal (Figure 3.3). Archdeacon proved that the list is complete and those are the only such graphs [1].

Chapter 3

Problem and current position of the field

Definition 3.1. *A graph G has a planar emulator (cover) H if H is a planar graph and there exists a graph homomorphism $\varphi : V(H) \to V(G)$ such that, for every vertex $v \in V(H)$, the neighbours of $v \in H$ are mapped by φ surjectively (bijectively) onto the neighbours of $\varphi(v)$ in G. The homomorphism φ is called an emulator (cover) projection.*

Informally, every vertex in G is represented by one or more vertices in its emulator H, and for every neighbour u of a vertex v in G, any vertex representing v in H has *at least one* (or *exactly one* in the case of a cover) neighbour representing u. See Figure 3.1 for examples.

We should point out that if there are no restrictions posed on the required planar emulator, it is always possible to construct an emulator in the form of an infinite tree. Such an emulator is, however, rather boring and thus, we are interested in the finite cases only.

The concept of finite planar emulators, first proposed by M. Fellows in 1985 [12], and the concept of finite planar covers by Negami [27] are tightly connected (despite their independent origin). In both cases, the main question is for which graphs it is possible to construct planar emulators and covers.

Conjecture 3.2 (Negami [27], 1988)**.** *A graph has a finite planar cover if and only if it embeds in the projective plane.*

Conjecture 3.3 (M. Fellows, Kitakubo, falsified 2008)**.** *Let G be a graph.*

(a) G has a finite planar emulator if and only if it has a finite planar cover.

(b) G has a finite planar emulator if and only if it embeds in the projective plane.

Negami conjectured that the class of planar-coverable graphs is identical to the class of projective graphs (see Conjecture 3.2). Fellows claimed, in

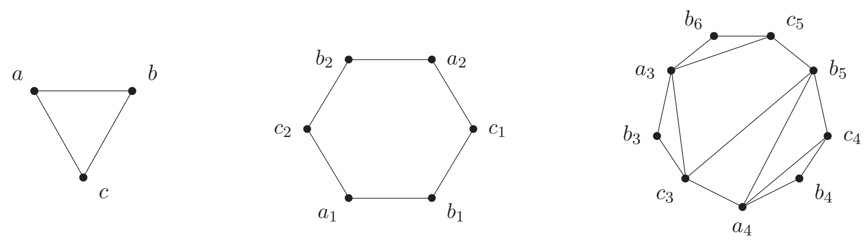

Figure 3.1: Examples of a planar cover (center) and a planar emulator (right) of the triangle $G = K_3$ (left). We simply denote by a_j, $j = 1, 2, \dots$ the vertices representing a of G, and analogously with b, c.

our opinion quite surprisingly, that both the classes of graphs with finite planar covers and those with planar emulators are identical (cf. Conjecture 3.3(a)). Believing that Negami's conjecture holds true, Fellows' claim was later reformulated by Kitakubo into Conjecture 3.3(b)). Note that while planar cover conjecture is still open and believed to be true, both claims of Conjecture 3.3 were already disproved (see Theorem 3.13).

The concept of planar covers was widely investigated by S. Negami and D. Archdeacon in the 1980's, and later by R. Thomas and P. Hliněný. In [26], it has been shown that all projective graphs are planar-coverable, and the construction of their covers is actually very simple — the crosscap in a projective-planar drawing of a graph G is replaced with a mirror image of the drawing, and the corresponding connections between the drawings are added (see Figure 3.2). It is obvious that every cover is also an emulator. Hence, Theorem 3.4 immediately follows.

Theorem 3.4. *If graph G has a projective embedding, it also has a finite planar emulator.*

To prove Conjectures 3.2 and 3.3, it would be necessary to show also the converse, i.e. that non-projective graphs are not planar-coverable and planar-emulable respectively. If G is a graph with finite planar emulator H, contraction or deletion of any edge uv in G can be performed as contraction or deletion of all edges induced by $\varphi^{-1}(u), \varphi^{-1}(v)$ in H without affecting

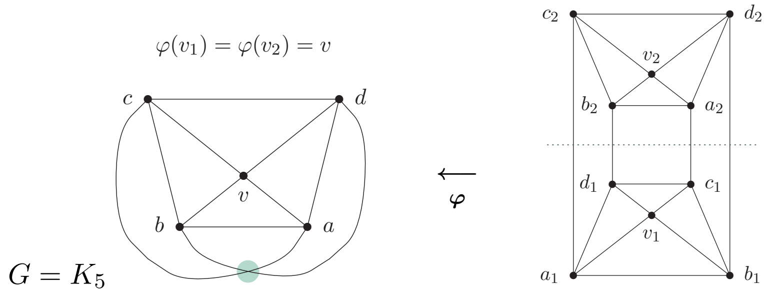

Figure 3.2: The graph $G = K_5$ (left) and its two-fold planar cover (right) via a homomorphism φ. The cover is obtained from a "crosscap-less" drawing of G and its mirror image.

planarity of H. Vertices can be deleted in a similar fashion. Thus, Proposition 3.5 easily follows.

Proposition 3.5. *The existence of finite planar emulators, and analogously also planar covers, is enclosed under taking minors.*

Hence to prove the conjectures, we can restrict ourselves to showing that there are no finite planar covers and emulators for the minor minimal obstructions for the projective plane, which are known [1]. This minor minimal family is depicted by Figure 3.3.

There are some structural dependencies that can be observed among the forbidden minors for the projective plane and that are actually reflected in planar covers and emulators. Consider the operation of replacing a cubic vertex by a triangle on its neighbours. This operation is called a *YΔ transformation*, the inverse is denoted by ΔY. The graphs $K_7 - C_4$, $\mathcal{D}_3$ and $\mathcal{E}_5$ can be obtained by a sequence of YΔ transformations from the graph $\mathcal{F}_1$. Similarly, the graphs $K_{1,2,2,2}$, $\mathcal{B}_7$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{D}_2$ can be obtained via YΔ transformations from the graph $\mathcal{E}_2$. Thus, the sets $\Delta\mathrm{Y}(K_7 - C_4) = \{K_7 - C_4, \mathcal{D}_3, \mathcal{E}_5, \mathcal{F}_1\}$ and $\Delta\mathrm{Y}(K_{1,2,2,2}) = \{K_{1,2,2,2}, \mathcal{B}_7, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2\}$ are called the families of $K_7 - C_4$ and $K_{1,2,2,2}$ respectively.

Proposition 3.6. *Existence of finite planar covers and emulators is enclosed under taking YΔ transformations.*

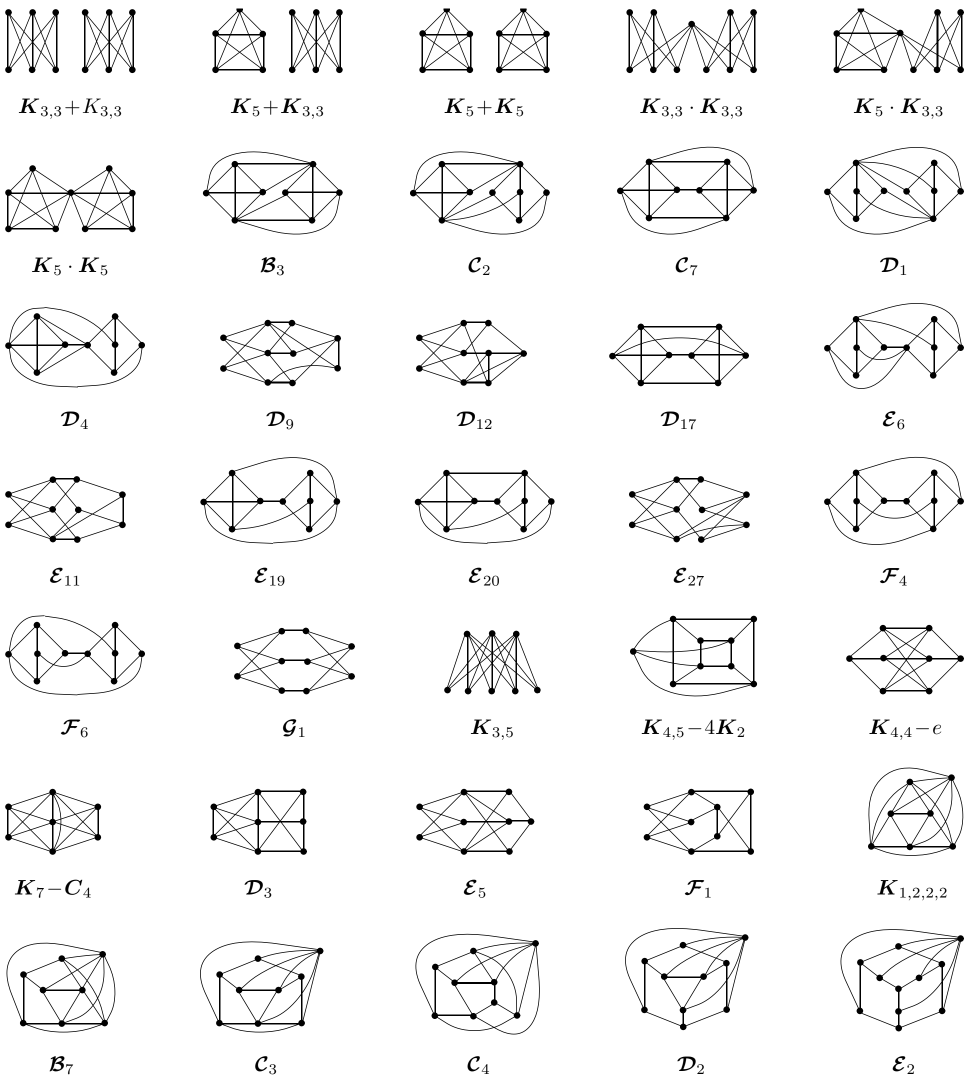

Figure 3.3: The minor minimal obstructions for the projective plane.

A brief idea why Proposition 3.6 holds can be obtained from Theorem 3.7 — it is possible to emulate independent cubic vertices with vertices of the same degree and thus, the triangle on the neighbours in the emulator can always be made. Theorem 3.7 is also a fundamental concept of the proof of Theorem 3.11. We remark that Theorem 3.7 was originally contained in [13], but, to the best knowledge of the author, it has not been published until [4].

Theorem 3.7 (M. Fellows, [13])**.** *Let G be a planar-emulable graph and $X \subseteq V(G)$ an independent set of cubic vertices (vertices of degree 3). Then there exists a planar emulator H of G with a projection $\varphi : V(H) \to V(G)$ such that every vertex $u \in \varphi^{-1}(v)$ over all $v \in X$ is of degree 3.*

As for Conjecture 3.2, proofs of non-existence of planar covers for some of the forbidden minors for the projective plane were provided (see Theorem 3.8). In particular, we would like to point out the non-existence of a finite planar cover for the graph $K_{4,5} - 4K_2$ [18].

Theorem 3.8 (Archdeacon, Fellows, Hliněný, and Negami, 1988-98)**.** *Conjecture 3.2 holds true if and only if there is no finite planar cover for the graph $K_{1,2,2,2}$.*

One can naturally think of applying the same arguments, which were used in the case of planar covers, to planar emulators, i.e. to Conjecture 3.3. This however cannot be done as all the more sophisticated tools (structural and discharging arguments) used to show non-existence of planar covers fail here (on a technical rather than conceptual level). In the following, we provide a review of the known obstructions for the finite planar emulators together with references to the particular proofs.

Definition 3.9. *Graph G contains two disjoint k-graphs if there exist two vertex-disjoint subgraphs $J_1, J_2 \subseteq G$ such that, for $i = 1, 2$, the graph J_i is isomorphic to a subdivision of K_4 or $K_{2,3}$, the subgraph $G - V(J_i)$ is connected and adjacent to J_i, and contracting all the vertices of $V(G) \setminus V(J_i)$ in G into one results in a non-planar graph (i.e. contracting a K_5- or $K_{3,3}$-subdivision).*

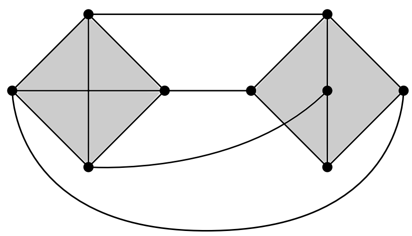

Figure 3.4: An example of a graph having two disjoint k-graphs (shaded in gray).

See Figure 3.4 for an example of a graph G with two disjoint k-graphs. Note that such G is also always non-projective [16].

Theorem 3.10 (Fellows [13]). *A planar-emulable graph G cannot contain two disjoint k-graphs. Consequently, none of the 19 graphs – projective forbidden minors – inbetween the graph $K_{3,3} \cdot K_{3,3}$ and $\mathcal{G}_1$ (incl.) of Figure 3.3 has a finite planar emulator.*

Theorem 3.11 (M. Fellows, [13]). *The graph $K_{3,5}$ has no finite planar emulator.*

The proofs of Theorems 3.10 and 3.11 can be studied e.g. from [4].

Let us remark that also by Euler's formula, graphs K_7 and $K_{4,4}$ cannot have finite planar emulators either. The following corollary is implied:

Corollary 3.12. *None of the graphs in the family $\Lambda = \{K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$, $K_{3,5}\}$ (see Figure 3.3) has a finite planar emulator. The graphs $K_{4,4}$ and K_7 are not planar-emulable either.*

The following result was a big breakthrough in the field of planar emulations and indeed, it falsified Conjecture 3.3.

Theorem 3.13 (Rieck and Yamashita [30], 2008). *The graphs $K_{1,2,2,2}$ and $K_{4,5} - 4K_2$ do have finite planar emulators.*

Theorem 3.13 is especially significant for the discovery of a finite planar emulator for the graph $K_{4,5} - 4K_2$ (Figure 3.5), which, as emphasized above,

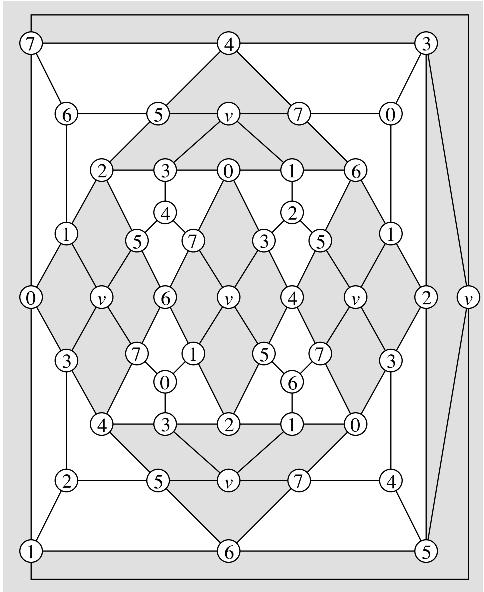

Figure 3.5: Planar emulator of the graph $K_{4,5} - 4K_2$ by Rieck and Yamashita.

is not planar-coverable. We remark that the emulator of $K_{4,5} - 4K_2$ is no longer a unique finding in this respect and finite planar emulators for the graph $\mathcal{E}_2$, $\mathcal{C}_4$ and all the members of $\Delta\mathrm{Y}(K_7 - C_4)$, which do not have finite planar covers either, have been found [4]. Consequently, the class of planar-emulable graphs is much larger than the class of planar-coverable graphs.

From Corollary 3.12 and Proposition 3.5, we obtain:

Corollary 3.14. *If a non-projective graph has a finite planar emulator, it must contain a minor isomorphic to one of $K_{4,4}-e$, $K_{4,5} - 4K_2$ or to a graph from the $K_7 - C_4$ or $K_{1,2,2,2}$ families.*

We remark that finite planar emulators do actually exist [4] for all the graphs in Corollary 3.14 with a single possible exception of $K_{4,4}-e$.

3.1 Role of internal 4-connectivity and methodology

A graph G is *internally 4-connected* if it is simple, 3-connected, has at least five vertices, and for every separation (A, B) of order 3, either $G \restriction A$ or $G \restriction B$ has at most three edges.

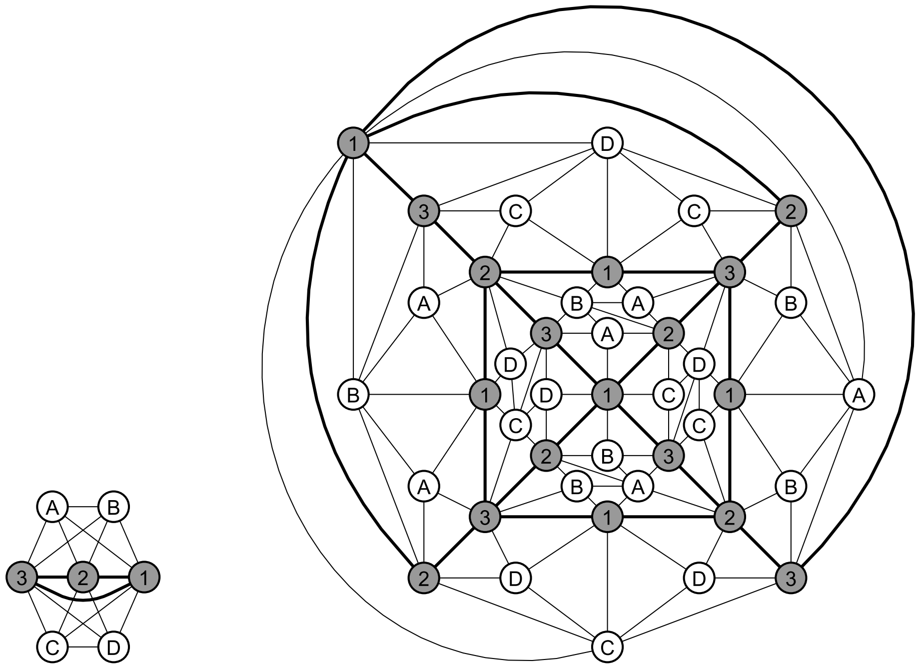

Figure 3.6: Planar emulator of the graph $K_7 - C_4$ [4].

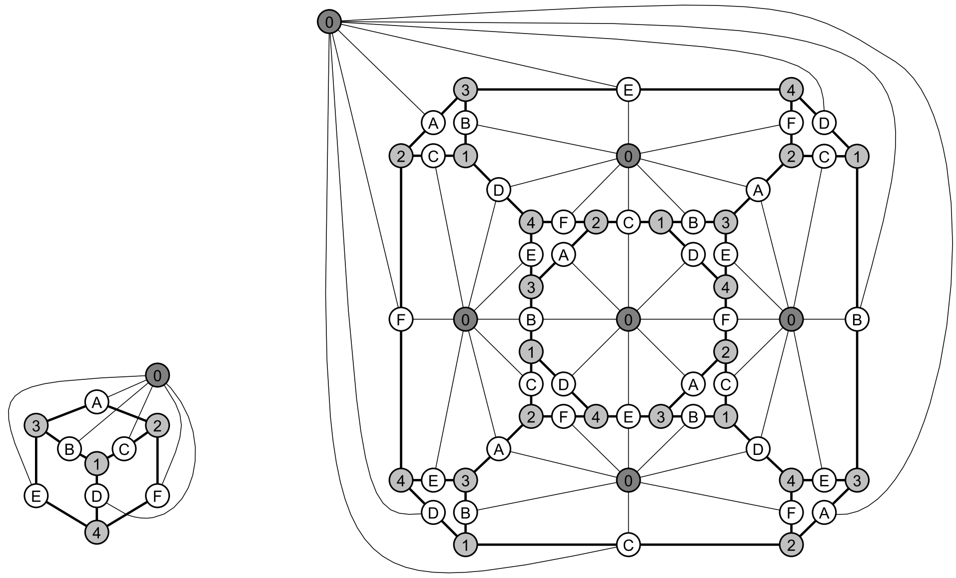

Figure 3.7: Planar emulator of the graph $\mathcal{E}_2$ from which emulators for the rest of graphs in the family of $K_{1,2,2,2}$ (except for $\mathcal{C}_4$) can be derived [4].

Let G be a graph. Let F be a connected planar graph on the vertex set $V(F)$ disjoint from $V(G)$, and let $x_1 \in V(F)$. If y_1 is a vertex of G, and the graph H_1 is obtained from $G \cup F$ by identifying the vertices x_1 and y_1, then H_1 is called a *1-expansion* of G. Let $x_1, x_2 \in V(F)$ be two distinct vertices that are incident with the same face in a planar embedding of F. If $e = y_1y_2$ is an edge of G, and the graph H_2 is obtained from $(G - e) \cup F$ by identifying the vertex pairs (x_1, y_1) and (x_2, y_2), then H_2 is called a *2-expansion* of G. Let $x_1, x_2, x_3 \in V(F)$ be three distinct vertices such that $F - \{x_1, x_2, x_3\}$ is connected. Moreover, let each of the vertices x_1, x_2, x_3 be adjacent to some vertex of $V(F - \{x_1, x_2, x_3\})$, and let all three vertices x_1, x_2, x_3 be incident with the same face in a planar embedding of F. If w is a cubic vertex of G with the neighbours y_1, y_2, y_3, and the graph H_3 is obtained from $(G-w) \cup F$ by identifying the vertex pairs (x_1, y_1), (x_2, y_2) and (x_3, y_3), then H_3 is called a *3-expansion* of G.

Definition 3.15 (Planar expansion). *A graph H is a planar expansion of a graph G if there is a sequence of graphs $G_0 = G, G_1, \ldots, G_l = H$ such that G_i is a 1-, 2- or 3-expansion of G_{i-1} for all $i = 1, \ldots, l$.*

The following is easy to see:

Lemma 3.16. *Let H be a planar expansion of G.*

(a) G has an embedding in the projective plane if and only if so does H.

(b) G has a finite planar emulator (cover) if and only if so does H.

(c) G is a minor of H.

The discovery of finite planar emulators for the graphs $K_{4,5} - 4K_2$ and $K_{1,2,2,2}$ (cf. Theorem 3.13) showed that there are non-projective graphs with finite planar emulators, but we do not know anything about this class. Thus, it is natural to ask how many such graphs exist and how large and complicated they can be.

In [20], Hliněný and Thomas provided the answer for a similar question for planar covers. They showed that all the minor-minimal non-projective graphs that possibly have a finite planar cover (recall that we do not know of any finite cover for a non-projective graph – Conjecture 3.2 is still open) are, up to planar expansions, internally 4-connected. Consequently, they searched for all such graphs and showed that there are at most 16 possible counterexamples to Conjecture 3.2.

The topic of finite planar emulators is widely discussed in the author's Bachelor's thesis [6]. We adopted the approach of Hliněný and Thomas and with sketch of a proof, we proposed that the non-projective graphs with finite planar emulators must be planar expansions of an internally 4-connected graph unless they contain a minor isomorphic to a member of the $K_7 - C_4$ family.

Theorem 3.17. *Let G be a connected graph that has a finite planar emulator, but no projective embedding. Then, G has a minor isomorphic to a member of the $K_7 - C_4$ family, or G is a planar expansion of an internally 4-connected graph (see below for the definitions).*

The proof of Theorem 3.17 is presented in Chapter 4.

Using the tools presented in [18] and later in this section, we conducted an exhaustive search described in [6]. As our research uses and builds up on the previous work, we are going to review the used methods now.

For the purpose of the following, consider this definition of a *vertex split* operation. Let a simple graph G be obtained from H by contracting an edge $e = uv \in E(H)$ to a vertex v. If degrees of the endvertices of e in H are at least 3, then H is said to be obtained from G by splitting the vertex v. The graph G is formally denoted by $G < v\left\{{N_1 \atop N_2}\right\}$, where N_1, N_2 are the neighbourhoods of endvertices u, v of e in H, respectively, excluding u, v themselves.

A graph G is called *almost internally 4-connected* is it is simple, 3-connected, has at least five vertices, and for every separation (A, B) of order 3, either $G \upharpoonright A$ or $G \upharpoonright B$ has at most four edges. The notion of almost internal 4-connectivity clearly differs from internal 4-connectivity by one edge only. Hence, a pair (v, e) where v is a cubic vertex in G and e has both the endvertices adjacent to v is called a *violating pair* in G. The edge e in violating pair (v, e) is referred to as a *violating edge*.

Given a violating edge $e = \{u_1, u_2\}$ in a simple graph G, the operation of a *triad addition* is defined as follows. Let v be a vertex of G distinct from

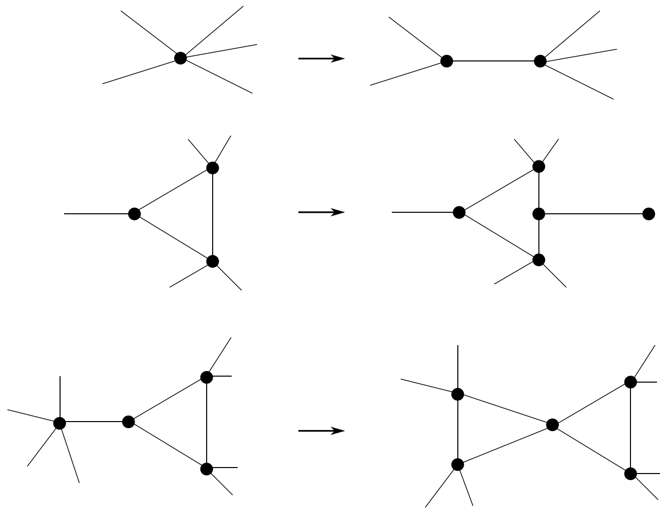

Figure 3.8: The operations of splitting a vertex, triad addition and triangle explosion.

and not adjacent to any of u_1 or u_2, and there is no violating pair (w, e) such that w and v are adjacent, the triad addition produces graph G_t from G by subdividing the edge e with a new vertex v' and connecting v to v' by an edge.

Let (w, e) be a violating pair in a simple graph G and u of degree at least 5 be the neighbour of w which is not incident with e. Then, the *triangle explosion* produces graph G_x from G by splitting the vertex u into vertices u_1, u_2 and by adding the missing one of edges $\{w, u_1\}$, $\{w, u_2\}$ so that the degrees of both u_1 and u_2 in the resulting graph are at least 4.

See Figure 3.8 for the operations introduced above.

Theorem 3.18, along with subsequent Theorems 3.19 and 3.20, are our main tools for producing internally 4-connected graphs.

Theorem 3.18 (T. Johnson, R. Thomas, 1997)**.** *Let G be an internally 4-connected minor of an internally 4-connected graph H such that G has no embedding in the projective plane. Then there exists a sequence $J_0 = G, J_1, \dots J_k \simeq H$ of almost internally 4-connected graphs such that for $i = 1, 2, \dots k$, the graph J_i is obtained from J_{i-1} by adding an edge, splitting a vertex, or by a triad addition or*

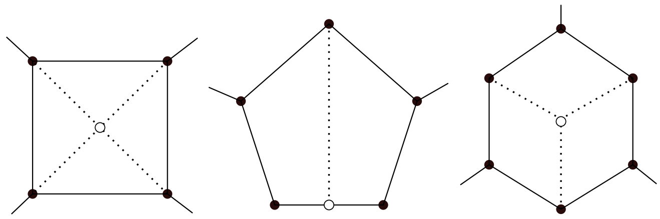

Figure 3.9: Quadrangular, pentagonal and hexagonal extension.

by a triangle explosion. Moreover, each J_i has at most one violating edge and if an edge e is contained in both J_{i-1} and J_i, it is not violating in at least one of them.

Let us remark that no proof of Theorem 3.18 was published so far. It was used to generate results of Hliněný in [18] and our Bachelor's research. Despite it is believed to be correct, we suggest that it should be perceived rather as a useful tool for double checking our results presented in Chapter 6, which are based on Theorem 3.19 [1].

Let H be an internally 4-connected graph, let $t \geq 1$ be an integer, and let $H_0 = H, H_1, \dots, H_t$ be a sequence of graphs such that for $i = 1, 2, \dots, t$ the following conditions are satisfied:

(a) $H_i = H_{i-1} + \{u_i, v_i\}$, where u_i, v_i are two distinct non-adjacent vertices of H_{i-1}.

(b) No edge is violating in both H_{i-1} and H_i.

(c) If $1 < i < t$, then H_i has at most one violating pair.

(d) H_t is internally 4-connected.

Then we say that H_t is an *addition extension* of H. We also say that H_t is a *t-step addition extension* of H. Let us point out that in condition (c), we do mean $i > 1$, i.e. H_1 is permitted to have more than one violating pair (but it has at most one violating edge, because H is internally 4-connected).

1. Theorem 3.18 was used to generate graphs for the graph covering problem within [18]. It provided the exact same results as the ones published later in [20] which were obtained by application of Theorems 3.19 and 3.20.

Let G be a graph, let $\{u, v, x, y\}$ be cubic vertices forming a cycle of length 4 in G. Then a graph H obtained by adding a vertex w to G and connecting it by an edge to all of $\{u, v, x, y\}$ is called a *quadrangular extension* of G, denoted by $H = G \boxtimes \{u, v, x, y\}$.

Let G be a graph and $(v_1, v_2, v_3, v_4, v_5)$ a cycle C of length 5 in G (with vertices in this order). Assume that v_2 and v_5 are cubic vertices and that v_1 is not adjacent to any of v_3 or v_4, and let e denote the edge $\{v_3, v_4\}$ of C. Then a graph H obtained by subdivision e by a vertex w and connecting w to v_1 by an edge is called a *pentagonal extension* of G, denoted by $H = G \pentagon \{v_1, v_2, v_3, v_4, v_5\}$.

Let G be a graph and $\{u, v, w\}$ independent vertices in G. Furthermore, assume that no cubic vertex of G has all the neighbors u, v, w and that every pair of vertices from $\{u, v, w\}$ have a common neighbor of degree three. In such circumstances, we say a graph H obtained by adding a new vertex x into G and connecting x to all of $\{u, v, w\}$ by an edge is a *hexagonal extension* of G, denoted by $H = G \hexagon \{u, v, w\}$.

The following theorem is a simplified version of a result proved in [21]:

Theorem 3.19 (T. Johnson, R. Thomas [21])**.** *Suppose that G and H internally 4-connected graphs, G is a proper minor of H, and that G has no embedding in the projective plane. Then, there exists a minor H' of H satisfying one of the following: H' is a t-step addition extension of G, or H' is a quadrangular, pentagonal or hexagonal extension of G, or H' is obtained by splitting a vertex.*

Note that unlike the other cases, if the graph H' is obtained from G by splitting a vertex, it is not necessarily internally 4-connected. In our search, we solve this by continuous removing the violating edges of such H' until we obtain an internally 4-connected subgraph of H', also minor of H.

In [21], Theorem 3.20 is provided as a stronger version of Theorem 3.19.

Theorem 3.20 (T. Johnson, R. Thomas [21])**.** *Suppose that G and H internally 4-connected graphs, G is a proper minor of H, and that G has no embedding in the projective plane. Assume further that each component of the subgraphs of G induced by cubic vertices is a tree or cycle. Then, either H is an addition extension of G, or there exists a minor H' of H satisfying one of the following: H' is a 1-*

step addition extension of G, or H' is a quadrangular, pentagonal or hexagonal extension of G, or H' is obtained by splitting a vertex.

We refer to Theorem 3.20 as stronger since, as long as the additional assumption about cubic vertices in G is satisfied, it allows us not to consider the operations other than addition extensions. To be more precise, assume non-projective graphs F, G, H such that F is a minor of G and G is a minor of H (so does obviously F). Assume further that G is a t-step addition extension of F with $t > 1$ and no subgraph of G is a 1-step addition extension of F. Then, H must be a t'-step addition extension of G, or there exists a minor H' of H that is distinct from G and that can be obtained from F by the four listed operations. Consequently, we do not need to take into account the graphs that can be obtained from G by operations other than t'-step addition extensions.

Within our Bachelor's research, we used Theorem 3.18 to generate all the non-projective internally 4-connected graphs that do not contain a minor from the family of $K_7 - C_4$ and can have finite planar emulator. By Corollaries 3.14 and 3.12, such graphs contain a minor from the family of $K_{1,2,2,2}$, $K_{4,5} - 4K_2$ or $K_{4,4}{-}e$ and do not contain 2 disjoint k-graphs or a minor isomorphic to $K_{3,5}$. The family of $K_7 - C_4$ was excluded and postponed for future work as these minor minimal obstructions are not internally 4-connected and thus Theorem 3.18 is not applicable. Furthermore, at that point, planar-emulability of $\Delta\mathrm{Y}(K_7 - C_4)$ was open and we hoped in having a proof that the emulators exist rather than finding some. Recall that the emulators for the family of $K_7 - C_4$ are known [4].

Our previous research showed the following theorem:

Theorem 3.21. *If H is a non-projective graph with finite planar emulator, then H is a planar expansion of an internally 4-connected graph G from a finite set of 176 graphs, or it contains a minor isomorphic to $K_{4,5} - 4K_2$, $\mathcal{E}_2$ or to a graph from the family of $K_7 - C_4$.*

Despite our enormous effort, we did not manage to generate all the internally 4-connected graphs with $\mathcal{E}_2$ or $K_{4,5} - 4K_2$ minors. This was given by the necessity of searching for various minor obstructions for planar emulators in generated graphs and by the large number of graphs produced by Theorem 3.18.

Given above, the challenges and main goals of this work are following:

(1) To provide a proper proof of Theorem 3.17. The proof is presented in Chapter 4.

(2) To reproduce the previously generated results graphs using Theorems 3.19 and 3.20 proofs of which were publish (unlike previously used Theorem 3.18). Results of this part of work were already published in [7] and are presented in Chapter 6.

(3) To finish computations for graphs $\mathcal{E}_2$ and $K_{4,5} - 4K_2$. This requires adjusting the current tool and their optimization, especially the algorithm used for searching for forbidden minors. Theorems 3.19 and 3.20 might be more suitable for this task. We managed to finish all the computations for the graph $\mathcal{E}_2$. The results are presented in Section 6.6. Note that these results were not included in [7].

(4) To analyse the obtained result and suggest further steps heading towards characterization of planar-emulable graphs. The analysis is available in 6.8 and further directions provided in Chapter 8.

We provide a detailed description of our approach to the generating process and optimization that our tools use in Chapter 5. In addition in Chapter 7, we tackle the problem of planar-emulations restricted to the class of cubic graphs only.

Chapter 4

Proof of Theorem 3.17

In this chapter, we are going to present a proof of Theorem 3.17. Let us point out that we have already published this part of our work in [7], however due to a discovered mistake in the picture of graph as $\mathcal{E}_{11}$ presented in [18, Appendix A] (and consequently in [20, 7, 4]), the proof in this chapter contains a minor correction (with respect to previous [7]). However, the main idea and core of the proof remains unchanged.

We remark that Theorem 3.17 will likely follow quite easily from a new approach to Archdeacon's result [1] which is currently being prepared by Asadi, Postle and Thomas [2], but in the meantime we present our independent arguments. Before we approach to indeed a very technical proof of this statement let us introduce the notion used in this section properly.

Suppose that G is a graph and $v_1, v_2, v_3 \in V(G)$ are three distinct vertices of G. Let *3-extension* of G be the graph H defined as follows: If there are two or more common cubic neighbours of v_1, v_2, v_3 then $H = G$. If there exist a cubic vertex $w \in V(G)$ adjacent to v_1, v_2, v_3 then H results from G by adding one new vertex t adjacent to all three vertices v_1, v_2, v_3. Otherwise, H results from G by adding two new vertices s, t both adjacent to all three vertices v_1, v_2, v_3. A *reduced 3-extension* of G is the graph H_0 obtained from a 3-extension H by removing possible edge between v_1, v_2, v_3. Notice that presence of such edges does not influence the embeddability and emulability properties, cf. Proposition 3.6.

From now on, we can start building the theory leading to the proof of Theorem 3.17 which is our main task in this section. For purpose of the rest of this section, let $\Lambda = \{K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$, $K_{3,5}\}$ be the family of minor minimal non-projective graphs without planar emulators (cf. Corollary 3.12).

A separation (A, B) in G is called *flat* if the graph $G \upharpoonright B$ has a planar embedding with all the vertices of $A \cap B$ incident with the outer face.

Lemma 4.1 (Hliněný, Thomas [20]). *Let G be a 3-connected graph, and let (A, B) be a non-flat separation of order three in G. Let F_0 be a simple 3-connected graph. Suppose that $F \subseteq G$ is a subgraph of G isomorphic to a subdivision of F_0, and that $W \subseteq V(F)$ is the subset of vertices that have degree more than 2 in F. If $|W \cap (B - A)| \leq 1$, then G contains a minor isomorphic to a 3-extension of F_0.*

Lemma 4.2. *Let G be a 3-connected graph, and let (A, B) be a separation of order three such that neither of (A, B) or (B, A) is flat. Assume H is a minor of G such that H is an internally 4-connected minor-minimal non-projective graph (Figure 3.3). Then, G contains a minor isomorphic to a reduced 3-extension of H, or G contains another minor F having two disjoint k-graphs or isomorphic to $K_{3,5}$ or a member of $\Delta Y(K_7 - C_4)$.*

Proof. First, note that since H is an internally 4-connected minor of G and both (A, B) or (B, A) are non-flat, then the condition $|W \cap (B - A)| \leq 1$ given by Lemma 4.1 is satisfied and G contains a minor isomorphic to a 3-extension of H', where H is a minor of, up to violating edges, internally 4-connected graph H'. If H' is obtained from H, by adding edges, then G contains a minor isomorphic to a 3-extension of H and this case is not interesting. Hence H' is obtained from H only by splitting vertices.

From Lemma [20, Lemma 3.4(b)] follows that if H is one of the graphs $K_{1,2,2,2}$, $\mathcal{B}_7$, $\mathcal{C}_3$ or $\mathcal{D}_2$ and G is obtained from H by vertex splits, then G contains a minor isomorphic to a member of Λ, $\Delta Y(K_7 - C_4)$ or to one of the graphs $K_{4,4} - e$, $K_{4,5} - 4K_2$, $\mathcal{C}_4$ or $\mathcal{E}_2$. Hence, we need to focus on the graphs $K_{4,4} - e$, $K_{4,5} - 4K_2$, $\mathcal{C}_4$ and $\mathcal{E}_2$ only.

So, let graph H' be obtained from H by a sequence of splitting a vertex v in H into vertices $v, w_1, \ldots, w_k$ in H', and let $H'_E \lessdot \{v_1, v_2, v_3\}$ be a 3-extension of H'. If $|\{v_1, v_2, v_3\} \cap \{v, w_1, \ldots, w_k\}| \leq 1$ for every split vertex v, then H'_E contains a minor isomorphic to a reduced 3-extension of H (via backward contraction of $\{v, w_1, \ldots, w_k\}$ into v). This is clear unless H' contained a cubic vertex s adjacent to v_1, v_2, v_3 while no such vertex s exists in H. In the latter case, s will play the missing role in a reduced 3-extension of H.

If $\{v_1, v_2, v_3\} \subseteq \{v, w_1, \ldots, w_k\}$ for a split vertex v, then the two common cubic neighbours of $\{v_1, v_2, v_3\}$ in H'_E form a subgraph isomorphic to $K_{2,3}$, and the vertices in $H' - v$, as H' is non-planar, form the other part of two disjoint k-graphs in H'_E.

The last case, without loss of generality, is that $\{v_1, v_2\} \subseteq \{v, w_1, \ldots, w_k\}$, and $v_3 \in \{u, z_1, \ldots, z_k\}$, where $\{u, z_1, \ldots, z_k\}$ are the vertices created by splitting a vertex $u \in H$ distinct from v. In this case, there is a connection of u and v in G, which is not necessarily an edge of H. To apply the same argument about the existence of two disjoint k-graphs in H'_E as in the previous case, we need to show that every graph X obtained by an edge contraction or unifying two nonadjacent vertices from $H \in \{K_{4,4}-e, K_{4,5} - 4K_2, \mathcal{C}_4, \mathcal{E}_2\}$ is non-planar. We consider all the possible cases of X.

Let $H = K_{4,4}-e$, let us denote the two partitions of H by A, B and let $x_a \in A$ and $x_b \in B$ be the vertices of degree three in H. Without loss of generality, if $e = x_a y$ for any vertex $y \in V(X)$, then the graph X contains a $K_{3,3}$ subgraph induced by the six vertices of degree four in H. So, let $e = y_a y_b$ with $y_a \in A$, $y_b \in B$ distinct from x_a, x_b. Then, X contains a $K_{3,3}$ subgraph on vertex set $V(X) - x$ for all $x \in \{x_a, x_b\}$. The $K_{3,3}$ subgraph remains unchanged even after unifying vertices x_a, x_b. So, consider the graph X obtained by unifying $y_{1a}, y_{2a} \in A$, i.e. $X \simeq H - y_{1a}$. But in this case, X has a minor isomorphic to K_5 and hence, it is non-planar.

Consider the labeling of the graph $H = \mathcal{E}_2$ as in Figure 4.1. Let $\sigma_1 = \{0, 4, 6, 7, 8\}$, $\sigma_2 = \{1, 2, 5, 7, 9\}$ and $\sigma_3 = \{10\}$ be the sets, classes of equivalence, of the vertices that are symmetric to each other. Note that three vertices of σ_1 and three vertices of σ_2 form several subdivisions of $K_{3,3}$ in H. By contracting an edge between vertices $u \in \sigma_i$ and $v \in \sigma_j$ for $i, j \in \{1, 2, 3\}$, some of the subdivisions of $K_{3,3}$ always remain unchanged and hence, resulting X cannot be non-planar. As $|\sigma_1| = |\sigma_2| = 5$, even after unifying two vertices u, v within σ_1 or σ_2, the subdivision of $K_{3,3}$ is still present.

Let $H = \mathcal{C}_4$. In this case, we list only the ten non-symmetric choices of the edge to be contracted to obtain graph X (it is easy to verify that none of such graphs is planar). The edges, referring to the labeling introduced by Figure 4.1, are $\{0, 1\}$, $\{0, 6\}$, $\{0, 7\}$, $\{3, 6\}$, $\{2, 6\}$, $\{3, 8\}$, $\{4, 5\}$, $\{4, 6\}$ and $\{4, 7\}$. As for unifying the vertices, there are the following pairwise non-

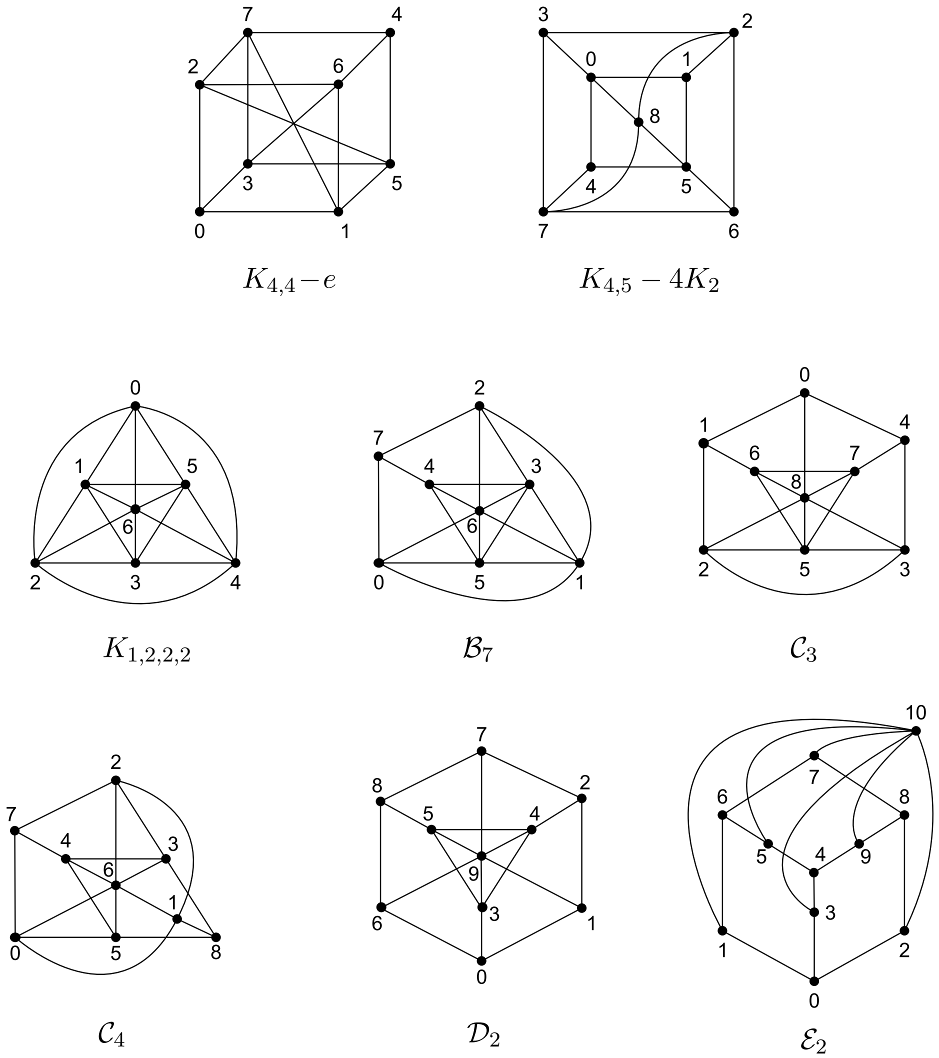

Figure 4.1: Illustration of the graphs $F_1 = K_{4,4}-e$, $F_2 = K_{4,5} - 4K_2$, $F_3 = K_{1,2,2,2}$, $F_4 = \mathcal{B}_7$, $F_5 = \mathcal{C}_3$, $F_6 = \mathcal{C}_4$, $F_7 = \mathcal{D}_2$ and $F_8 = \mathcal{E}_2$ introducing the labeling for the proof of Lemma 4.4. The drawings are different from the usual representations (cf. Figure 3.3) for the symmetries in the graphs to be more obvious.

symmetric cases:

- If X is obtained via unifying vertices $\{0, 2\}$, it contains K_5 minor via contracting the vertices $\{0, 2, 7\}$ and $\{1, 3, 8\}$.
- If X is obtained by unifying vertices $\{2, 8\}$ into vertex 2, it contains a subdivision of K_5 formed by the vertices $\{1, 2, 3, 4, 6\}$ and vertices 5 and 0 subdividing the connections from between $\{4, 2\}$ and $\{1, 6\}$ respectively.
- If X is obtained via unifying vertices $\{3, 5\}$, it contains K_5 minor via contracting $\{7, 4\}$ and $\{1, 2, 8\}$.
- If X is obtained by unifying vertices $\{3, 7\}$ into 3, it contains K_5 minor via contracting edges $\{1, 2\}$, $\{5, 8\}$ and $\{4, 6\}$.
- If X is obtained by unifying vertices $\{0, 3\}$ into 3, it contains K_5 minor via contracting edges $\{1, 2, 7\}$ and $\{5, 8\}$.
- If X is obtained by unifying vertices $\{2, 5\}$, it contains K_5 minor via contracting vertices $\{4, 7\}$ and $\{1, 3, 8\}$.

Hence, for all the possible graphs X for $H = \mathcal{C}_4$ are non-planar.

Let $H = K_{4,5} - 4K_2$ with labeling as in Figure 4.1. Let $\sigma_1 = \{0, 2, 5, 7\}$, $\sigma_2 = \{1, 3, 4, 6\}$ and $\sigma_3 = \{8\}$ be the set of pairwise symmetric vertices. Three vertices of σ_1 and three vertices of σ_2 form multiple subdivisions of $K_{3,3}$ in H. One of them is always preserved even after contracting an edge between σ_1 and σ_2 or unifying two vertices u, v within σ_1 or σ_2. The same argument applies for unifying a vertex in σ_3 with a vertex of σ_1.

Without loss of generality let the graph X be obtained from H by contracting the edge $\{1, 8\}$ into vertex 1. In this case, X is planar. However, we show that in this case, H'_E, and hence also G, contains another minor F isomorphic to $\mathcal{F}_1 \in \Delta\mathrm{Y}(K_7 - C_4)$.

There are two options of obtaining H' from H

- $H' = H < 8\left\{\begin{smallmatrix} v_1, v_2 \\ v_3, v_4 \end{smallmatrix}\right\}$ for $v_1, v_2, v_3, v_4 \in \sigma_1$. Due to symmetries within σ_1, all such H' are symmetric. Hence, let us choose $v_1 = 0$, $v_2 = 2$,

$v_3 = 5$ and $v_4 = 7$. Now, there are only two non-isomorphic cases for $H'_E = H' {<} \{8, 9, x\}$ for $x \in V(H')$
$0, 2$, where 9 is the vertex created by splitting 8. If $x \in \sigma_1$, then H'_E contains $\mathcal{F}_1$ subgraph via contracting $\{x_1, y_1, y_2\}$, where $x_1 \in \sigma_1$
x and y_1, y_2 are its two neighbours from σ_2, and $\{x_2, y_3\}$ with $x_2 \in \sigma_1$
x, x_1 and $y_3 \in \sigma_2$
y_1, y_2. If $x \in \sigma_2$, then H'_E contains $\mathcal{F}_1$ subgraph via contracting edges $\{x, y\}$, $\{x_1, y_1\}$, $\{x_2, y_2\}$, where $\{y, y_1, y_2\} \subseteq \sigma_1$ and $\{x_1, x_2\} \subseteq \sigma_2$. In both the cases, the extended vertices $\{8, 9, x\}$ create the $K_{2,3}$ in H'_E, the rest of the vertices form the remaining part of $\mathcal{F}_1$.

- $H' = H {<} 0\left\{ {8, v_1 \atop v_2, v_3} \right\}$, where v_1, v_2, v_3 are the neighbours of vertex 0, all of them belonging to σ_2. There are in total 6 pairwise non-isomorphic 3-extensions $H'_E {<} \{0, 9, x\}$, all of which contain a $\mathcal{F}_1$ subgraph via, without loss of generality, contracting edge $\{2, 3\}$ and
  - $\{0, 4\}$ and $\{1, 5\}$ for $x \in \{2, 3, 7\}$
  - $\{4, 7\}$ and $\{1, 5\}$ for $x = 1$
  - $\{4, 5, 7\}$ for $x = 6$
  - $\{1, 5, 6\}$ for $x = 8$

There are no other options. □

Lemma 4.3. *Let G be a connected graph that has no embedding in the projective plane, and that has no minor isomorphic to a member of Λ or $\Delta Y(K_7 - C_4)$. If $k \in \{1, 2, 3\}$ is the least integer such that there is a nontrivial separation (A, B) of order k in G, then either (A, B) or (B, A) is flat.*

Proof. For the cases of $k = 1$ and $k = 2$, we refer to the argumentation provided within the proof of [20, Lemma 2.2]. Since there are differences in argumentation for $k = 3$, we focus on this case.

So, let $k = 3$ and suppose, for a contradiction, that neither (A, B) nor (B, A) are flat. Notice that the assumptions guarantee that G is 3-connected in this case. By Corollary 3.14, G has a minor isomorphic to $M = F_i$ for

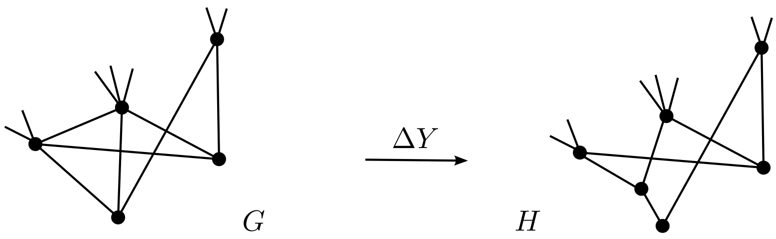

Figure 4.2: By replacing a triangle with a cubic vertex (ΔY transformation) in a 3-extension G, we obtain a subdivision of a reduced 3-extension H. As planar-emulability is enclosed under YΔ transformations, if G does not have a finite planar emulator, H does not either.

some $0 < i \leq 8$, where $F_1 = K_{4,4} - e$, $F_2 = K_{4,5} - 4K_2$, $F_3 = K_{1,2,2,2}$, $F_4 = \mathcal{B}_7$, $F_5 = \mathcal{C}_3$, $F_6 = \mathcal{C}_4$, $F_7 = \mathcal{D}_2$ and $F_8 = \mathcal{E}_2$. Let i be the maximum such integer. Since F_i is internally 4-connected and G does not contain any minor isomorphic to a member of Λ, we deduce from Lemma 4.1 that graph G contains a minor isomorphic to a reduced 3-extension of F_i. Thus, G contains a minor isomorphic to a member of Λ or $\Delta\text{Y}(K_7 - C_4)$, which is shown later as Lemma 4.4, a contradiction. $\square$

Lemma 4.4. *Let E be a reduced 3-extension of a graph F_i, where $F_1 = K_{4,4} - e$, $F_2 = K_{4,5} - 4K_2$, $F_3 = K_{1,2,2,2}$, $F_4 = \mathcal{B}_7$, $F_5 = \mathcal{C}_3$, $F_6 = \mathcal{C}_4$, $F_7 = \mathcal{D}_2$ and $F_8 = \mathcal{E}_2$, on vertices $v_1, v_2, v_3 \in V(F_i)$. Then, E contains a minor isomorphic to a member of Λ or $\Delta\text{Y}(K_7 - C_4)$.*

Proof. First, note that if a 3-extension E' of F_i contains a minor $M \in \Lambda \cup \Delta\text{Y}(K_7 - C_4)$, then also the reduced 3-extension $E \subseteq E'$ contains a minor M' isomorphic to M or to a graph that can be obtained from M by ΔY (inverse of YΔ) transformations (see Figure 4.2). Such M' must belong to $\Lambda \cup \Delta\text{Y}(K_7 - C_4)$ as well. Hence, we prove Lemma 4.4 by showing that all 3-extensions E' of all F_i with $1 \leq i \leq 8$ contain a minor isomorphic to a member of Λ or $\Delta\text{Y}(K_7 - C_4)$.

The proof for F_i with $i \in \{3,4,5,7\}$ is provided within the proof of [20, Lemma 2.2, p. 13]. Hence, we focus on the remaining cases, i.e. F_1, F_2, F_6 and F_8. Figure 4.1 depicts all the graphs F_i with emphasis on their symmetries.

Case $\mathbf{F_1 = K_{4,4} - e}$:

Let vertices 0 and 7 be the cubic vertices of $K_{4,4} - e$. It is easy to see that for every 3-extension of F_1 on triple of vertices $\{a,b,c\}$, one of the following situations applies:

- The triple $\{a,b,c\}$ does not contain any of the vertices $\{0,7\}$. Then, an edge $e = uv$ such that $u \in \{a,b,c\}$ and $b \in \{0,7\}$ can be contracted resulting in graph G that contains a subgraph isomorphic to $\mathcal{E}_5$.
- The triple $\{a,b,c\}$ contains just one of the vertices $\{0,7\}$. Then, the triple $\{a,b,c\}$ is symmetric to one of the following, and the corresponding construction can be applied:
  - for triple $\{0,3,6\}$, $\mathcal{E}_5$ subgraph is obtained via contracting the edge $e_1 = \{3,4\}$
  - for triple $\{0,2,6\}$, $\mathcal{E}_5$ subgraph is obtained via contracting the edge $e_2 = \{2,4\}$
  - for triple $\{0,2,4\}$, $\mathcal{E}_5$ subgraph is obtained via contracting the edge $e_3 = \{2,6\}$
- The triple $\{a,b,c\}$ contains both the vertices $\{0,7\}$. Then, an edge $f = uv$ such that $u \in \{0,7\}$ and $v \notin \{a,b,c\}$ can be contracted. The resulting 3-extension of F_i contains $\mathcal{E}_5$ subgraph.

Case $\mathbf{F_2 = K_{4,5} - 4K_2}$:

Consider the labeling for the graph F_2 as denoted by Figure 4.1. For every 3-extension $H = F_2{<}\{a,b,v\}$ with vertex v from the set $\{1,3,4,6\}$, one of the following constructions can be applied:

- For $v = 1$, either the edges $\{0,3\}$, $\{2,6\}$, $\{4,5\}$, or the edges $\{0,4\}$, $\{2,3\}$, $\{5,6\}$ are contracted.

- For $v = 3$, either the edges $\{0,4\}$, $\{1,2\}$, $\{6,7\}$, or the edges $\{0,1\}$, $\{2,6\}$, $\{4,7\}$ are contracted.

- For $v = 4$, either the edges $\{0,3\}$, $\{1,5\}$, $\{6,7\}$, or the edges $\{0,1\}$, $\{3,7\}$, $\{5,6\}$ are contracted.

- For $v = 6$, either the edges $\{1,2\}$, $\{3,7\}$, $\{4,5\}$, or the edges $\{1,5\}$, $\{2,3\}$, $\{4,7\}$ are contracted.

For all such 3-extensions H, the vertices v, a, b remain distinct after applying the contractions in at least one of the two cases mentioned above for each vertex v. The graph resulting from H by applying the contractions contains a subgraph S isomorphic to $K_5 - e$, where edge $e = \{a, b\}$. The vertex v is attached to a common neighbour of a, b in S – a vertex of degree four in S, by an edge. Hence, H must contain a $\mathcal{D}_3$ subgraph.

The 3-extensions $H = F_2 \lessdot \{a, b, c\}$ with a, b, c being a combination of $\{0, 2, 5, 7\}$ are symmetric to each other. Such vertices a, b, c already have a common cubic neighbour and therefore, only one is added to form the 3-extension H, which directly contains a subgraph isomorphic to $\mathcal{D}_9$.

The only remaining possibilities are the 3-extensions $H = F_2 \lessdot \{a, b, 8\}$ such that $a, b \in 0, 2, 7, 5$. Such 3-extensions are symmetric to each other. Without loss of generality, consider a 3-extension H of F_2 on the vertices $\{0, 2, 8\}$ with the edge $\{2, 3\}$ contracted. The resulting graph contains a subgraph isomorphic to $\mathcal{E}_{27}$.

Case $\mathbf{F_6} = \boldsymbol{\mathcal{C}_4}$:

There are in total 14 pairwise non-isomorphic 3-extensions $H = F_6 \lessdot \{a, b, c\}$ of $F_6 = \mathcal{C}_4$. Referring to the labeling introduced by Figure 4.1, they contain a $\mathcal{D}_3 \in \Delta\mathrm{Y}(K_7 - C_4)$ subgraph by contracting following components:

- $\{3, 4, 7\}$ and $\{1, 8\}$ for $\{a, b, c\} = \{0, 1, 2\}$

- $\{4,5\}$, $\{2,7\}$ and $\{1,8\}$ for $\{a,b,c\}=\{0,1,3\}$
- $\{3,4\}$, $\{0,7\}$ and $\{1,8\}$ for $\{a,b,c\}=\{0,2,6\}$
- $\{3,4,5\}$ and $\{1,8\}$ for $\{a,b,c\}=\{0,2,7\}$
- $\{3,4\}$, $\{0,7\}$ and $\{1,6\}$ for $\{a,b,c\}=\{0,2,8\}$
- $\{1,2\}$, $\{4,7\}$ and $\{5,8\}$ for $\{a,b,c\}=\{0,3,6\}$, $\{0,6,7\}$ or $\{0,6,8\}$
- $\{1,2\}$, $\{4,6\}$ and $\{5,8\}$ for $\{a,b,c\}=\{0,3,7\}$
- $\{1,2\}$, $\{4,7\}$ and $\{5,6\}$ for $\{a,b,c\}=\{0,7,8\}$
- $\{2,3\}$, $\{0,7\}$ and $\{5,8\}$ for $\{a,b,c\}=\{0,1,6\}$
- $\{2,3\}$, $\{4,7\}$ and $\{5,8\}$ for $\{a,b,c\}=\{0,1,7\}$
- $\{1,8\}$ and $\{4,5\}$ for $\{a,b,c\}=\{0,2,4\}$
- $\{0,1,2\}$ and $\{4,5\}$ for $\{a,b,c\}=\{6,7,8\}$

Case $\mathbf{F_8}=\boldsymbol{\mathcal{E}_2}$:

There are 16 pairwise non-isomorphic 3-extensions of $H=F_8{<}\{a,b,c\}$ of $F_8=\mathcal{E}_2$, which contain at least one of $\mathcal{F}_1,\mathcal{E}_5\in\Delta\mathrm{Y}(K_7-C_4)$ or $\mathcal{E}_{11},\mathcal{G}_1\in\Lambda$ minors as follows:

- For $\{a,b,c\}\in\{\{0,1,2\},\{1,2,3\},\{1,3,8\}\}$, a subgraph isomorphic to $\mathcal{F}_1\in\Delta\mathrm{Y}(K_7-C_4)$ is obtained via contraction of $\{4,5\}$, $\{6,7\}$, $\{0,3\}$ and $\{2,8\}$.
- For $\{a,b,c\}\in\{\{2,6,7\},\{2,6,8\},\{2,7,10\},\{2,8,10\}\}$, a subgraph $\mathcal{F}_1\in\Delta\mathrm{Y}(K_7-C_4)$ is obtained via contraction of $\{0,1\}$, $\{3,4\}$ and $\{7,8,9\}$.
- For $\{a,b,c\}\in\{\{3,4,7\},\{3,7,9\},\{3,7,10\}\}$, a subgraph $\mathcal{F}_1\in\Delta\mathrm{Y}(K_7-C_4)$ is obtained via contraction of $\{0,1\}$ and $\{2,4,8,9\}$.
- For $\{a,b,c\}=\{4,5,6\}$, a subgraph $\mathcal{F}_1\in\Delta\mathrm{Y}(K_7-C_4)$ is obtained via contraction of $\{0,1\}$, $\{2,8\}$ and $\{3,4,9\}$.

- For $\{a, b, c\} = \{5, 7, 9\}$, a subgraph $\mathcal{F}_1 \in \Delta\mathrm{Y}(K_7 - C_4)$ is obtained via contraction of $\{0, 1\}$, $\{6, 7\}$ and $\{8, 9\}$.
- For $\{a, b, c\} = \{2, 6, 10\}$, a subgraph $\mathcal{E}_5 \in \Delta\mathrm{Y}(K_7 - C_4)$ is obtained via contraction of $\{0, 1\}$, $\{3, 4\}$, $\{6, 7\}$ and $\{8, 9\}$.
- For $\{a, b, c\} = \{2, 6, 10\}$, a subgraph $\mathcal{E}_5 \in \Delta\mathrm{Y}(K_7 - C_4)$ is obtained via contraction of $\{0, 1\}$, $\{4, 5\}$ and $\{2, 8, 9\}$.
- For $\{a, b, c\} = \{4, 6, 8\}$, a subgraph $\mathcal{G}_1 \in \Lambda$ is obtained via contraction of $\{5, 7, 9, 10\}$.
- For $\{a, b, c\} = \{4, 6, 10\}$, a subgraph $\mathcal{E}_{11} \in \Lambda$ is obtained via contraction of $\{4, 5, 6\}$.

In all the cases listed above, the graphs contain a minor isomorphic to a member of $\Lambda \cup \Delta\mathrm{Y}(K_7 - C_4)$ and hence, Lemma 4.4 holds true. $\square$

Finally, we can approach the proof of Theorem 3.17.

Proof of Theorem 3.17. A corollary of Lemma 4.3 is that if G is a connected graph that has no embedding in the projective plane and that has no minor isomorphic to a member of Λ or $\Delta\mathrm{Y}(K_7 - C_4)$, then there exists an internally 4-connected graph H with the same properties, i.e. H is non-projective and has no minor isomorphic to a member of Λ or $\Delta\mathrm{Y}(K_7 - C_4)$, such that G is a planar expansion of H. See the proof of [20, Corollary 2.2] for the argumentation. Lemmas 4.3 and 3.16 respectively provide generalizations of [20, Lemma 2.2] and [20, Lemma 2.1] for finite planar emulators, which are referred to during the proof. Theorem 3.17 is then a simple reformulation of this corollary. $\square$

Chapter 5

Technical details and principles of generating

Our tools for conducting the exhaustive search went, up to this point, through three “stages of development”. The early stage tools were the ones that we used to initially explore the idea of the exhaustive search within our Bachelor’s research [6]. The tools were capable of applying Theorem 3.18 only and required a massively parallel environment[1] utilized when searching for forbidden minors. We did not manage to finish computations for the graphs $K_{4,5} - 4K_2$ and $\mathcal{E}_2$, which we gave up when processing graphs of size about 12 vertices only. These tools showed that Theorem 3.18 perhaps is not the splitter theorem that fits our problem best.

Therefore, already within our Master’s research, we approached adjusting our tools to be able to apply Theorem 3.19 and 3.20 as well. These adjustments gave rise to the second generation of our software, which was used to generate results for [7]. New splitter theorems not only proved themselves to offer much better performance than Theorem 3.18, but also provided a valuable verification of the previous results (see [7]). We still used parallel environment when searching for minors, but we incorporated some improvements based on previous experiences (e.g. searching for minors based on the origin of the graph). Despite our effort, we still have not managed to finish computations for the graphs $K_{4,5} - 4K_2$ and $\mathcal{E}_2$. We managed to process graphs of size up to 14 vertices generated from the aforementioned graphs. The second generation tools also revealed existence of what we now call violating minors and cycles in generating.

The lack of success in finishing the search for the two aforementioned graphs brought us to developing the latest generation of the search software. The main improvement was designing a beautifully simple, but extremely powerful, heuristic for searching for forbidden minors of our in-

1. Our computations were conducted on 64 Intel Xeon X7560 CPU’s at 2.27 GHz with hyperthreading

terest (see Section 5.5.1). This heuristic meant a significant performance improvement which allowed us to abandon the parallel environment for most of the graphs to be processed. In fact, the only graph that now requires parallelism is $K_{4,5} - 4K_2$. The remaining graphs, including $\mathcal{E}_2$ can be processed on a regular laptop[2]. Other than the minor search heuristic (see Section5.5) we integrated some other tools such as nauty [23] with our software. These tools were present in the previous development stages as well, however they were compiled separately and called by our software as external modules.

During this long process of development, we conducted many experiments striving for the best performance. Due to those experiments, we gained a lot of invaluable experiences with application splitter theorems to internally 4-connected non-projective graphs and the presented problem. The purpose of this section is to present some aspects of our approach to the exhaustive search. We are doing so in order to show that our computations and obtained results are not only correct, but, with a good level of insight, also fairly easy to reproduce.

Figures 5.1 through 5.4 document our use of Theorems 3.19 and 3.20 for generating internally 4-connected graphs. Figure 5.1 shows the overall algorithm. During the initialization step, the input — an internally 4-connected non-projective graph G, together with the general setup of generating process, e.g. what splitter theorem is to be used, is loaded.

The initialization also includes loading the following collections that are maintained during the search:

$$\begin{aligned} \varrho &\subseteq \mathcal{G} \\ \mathcal{A} &\subseteq \mathcal{G} \\ \mathcal{P} &\subseteq \mathcal{G} \\ \mathcal{L} &\subseteq \mathcal{G} \\ \mathcal{M} &\subseteq \mathcal{G} \times \mathcal{G} \times \{true, false\} \end{aligned}$$

where ϱ is the list of forbidden minors for graphs generated from G, $\mathcal{A}$ is the set of results — pairwise nonisomorphic internally 4-connected graphs

2. with 2.0 GHz Intel Core 2 Duo T7300 and 4GB RAM in our case

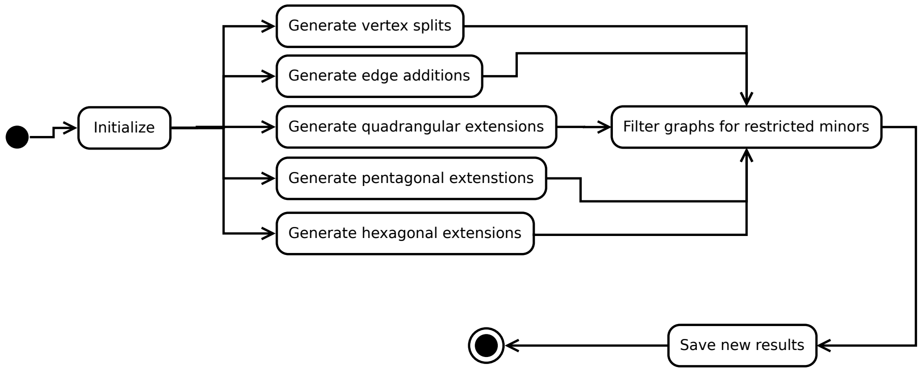

Figure 5.1: Scheme of generating according to Theorems 3.19 and 3.20.

generated so far (from all initial graphs, not only from G), $\mathcal{P}$ is the set of graphs for which the repairs of a violating edge were attempted before (see Section 3.1), and $\mathcal{M}$ is the set of positive or negative records $(H, M, bool)$ that represent already known information that graph H does or does not contain minor M. These collections are sorted in order to allow for fast searching. They are used for our computations to avoid redundant work. Their particular use is described in further sections within this chapter.

Then, all the graphs H that are t-step addition extensions of G are generated. The vertex splits, quadrangular, pentagonal and hexagonal extensions of G are generated as well if necessary (cf. Theorem 3.20). Every graph H produced in this phase is then passed to a routine that tests whether it contains a minor isomorphic to a graph from ϱ. The routine is described in detail in Section 5.5. If the graph contains a forbidden minor, it is rejected, otherwise it is passed to a saving module. This module verifies that this graph was not produced yet, i.e. it is not in $\mathcal{A}$, and if not, it is inserted in $\mathcal{A}$, saved and a request to start generating recursively using Theorem 3.19 or 3.20 is put through. If a violating minor is found, we also include another request to apply Theorem 3.18 to graph G.

5.1 Addition extensions

Having an internally 4-connected graph G, our use of the splitter theorems aims to generate all the internally 4-connected graphs H that are minimal, i.e. there is no other internally graph F with $G \leq F \leq H$.

More detailed depiction of generating addition extensions is provided within Figure 5.2. For an internally 4-connected graph $G = G_0$, first, all graphs G_1 that can be obtained by single edge addition are generated. For every G_1, the canonical representative is immediately computed, so only the non-isomorphic representatives are kept to continue with. Every graph then goes through a series of tests: the internally 4-connected graphs are designated for further processing, otherwise a process of repairs is initiated. Let G_i be a graph in the repair process. In an attempt to not compute anything more than once, we maintain a set $\mathcal{L}$ of graphs that already went through the repair process. Note that $\mathcal{L}$ is not a global collection passed through the entire search, it is initially empty for every internally 4-connected graph G that enters generating addition extensions. If graph G_i isomorphic to some already processed graph $F \in \mathcal{L}$ is to be repaired (such graphs can be created e.g. by adding edges in a different order), it is immediately rejected. Otherwise, we verify that G_i does not contain multiple violating edges or the same violating edge as the preceding graph G_{i-1}. Note that these conditions are not relevant for $i = 1$. If the graph G_i meets all the criteria and is not rejected, it is added to the list of repaired graphs and consequently, all graphs G_{i+1} are generated by single edge additions.

Before G_{i+1} is sent to the same repair routine, the very last condition is verified. In order to break the entire search into as many independent steps as possible, our algorithm attempts to construct all t-step addition extensions $H = G_t$ of $G = G_0$ such that there is no internally 4-connected addition extension G_j for $0 < j < t$. Thus, all graphs $G_j = (V(G_{i+1}), E(G_0) + S)$ such that S is a nontrivial subset of $E(G_{i+1}) \setminus E(G_0)$ are generated. If there is an internally 4-connected G_j among them, G_{i+1} is rejected as it will be obtained as a t-step addition extension of G_j later.

We tested many approaches to producing addition extensions. The one presented above provides by far the best performance we were able to achieve.

The process terminates when there are no more graphs to be repaired. The fact that the algorithm converges and is correct is obvious.

5.2 Vertex splitting

If vertex splits are required, the algorithm generates all the graphs that can be obtained by splitting a vertex u into vertices u_1, u_2. Splits with any of u_1, u_2 having degree less than 3 are omitted. Let G_1 denote a graph produced from an internally 4-connected $G = G_0$ by the described operation. Unlike graphs produced by quadrangular, pentagonal and hexagonal extension, G_1 does not need to be internally 4-connected in this case. The theorem does not provide any further information about such graphs, thus we approach iterative removal of the violating edges in order to create a sequence of graphs $G_2, G_3, \ldots, G_k, \leq$. We remove the edges as they are encountered by iterators used in the implementation, i.e. we do not enforce any specific order. Lemma 5.1 implies that if there is an internally 4-connected subgraph $F \leq G_1$ that can be obtained by such a process, the order in which the violating edges are removed does not matter.

Lemma 5.1. *Let $S \subseteq E(G)$ be the set of violating edges of a graph G_i. If there are two distinct violating edges $e_1, e_2 \in S$ such that e_2 would not be violating in $G_{i+1} = (V(G_i), E(G) \setminus \{e_1\})$, then G_{i+1} is not 3-connected.*

Proof. If edge e_2 of violating pair (v, e_2) is no longer violating after removing edge e_1, then the neighbours of vertex v no longer form a 3-cut. By removing an edge from the graph, the degree of vertex v could not increase, thus v is a vertex of degree 2 in G_{i+1}. $\square$

The order of violating edges to be removed matters if consecutive removing of violating edges produces $G_i \leq G_k$ that is not 3-connected. As discussed in the other sections, such situations do occur; we refer to such a vertex split as *violating split*. If a violating split is encountered, our software

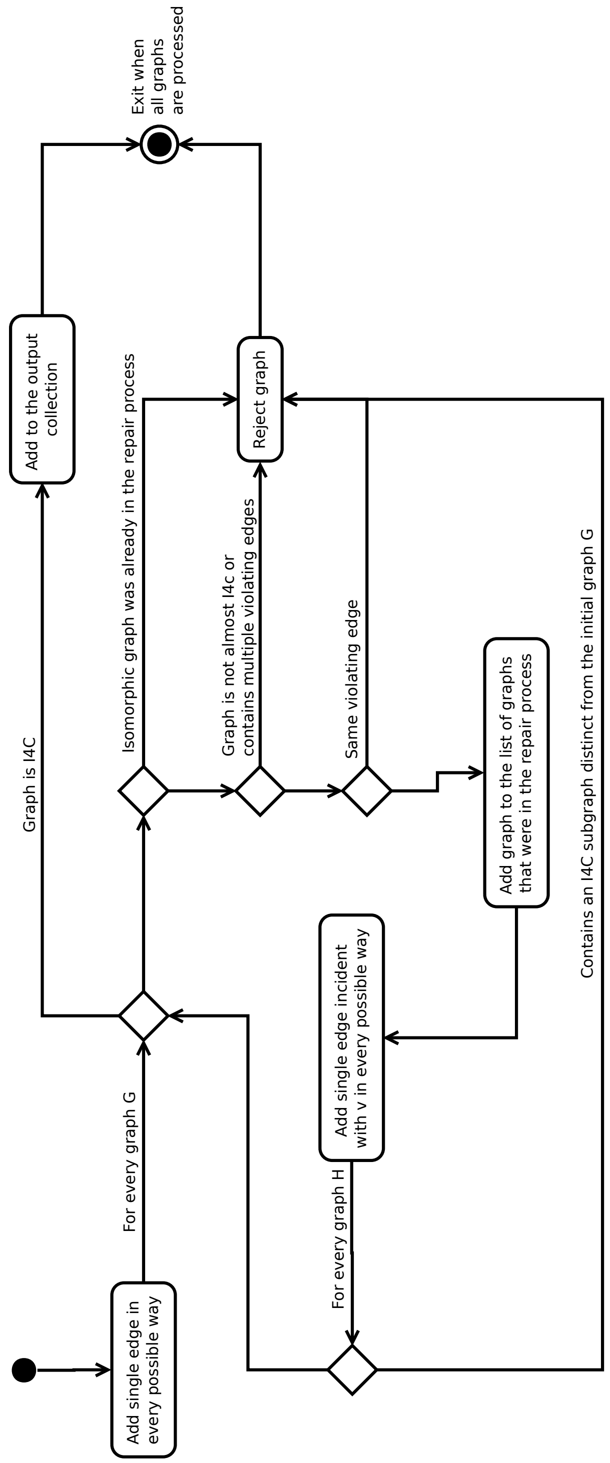

Figure 5.2: Scheme of generating t-step edge additions.

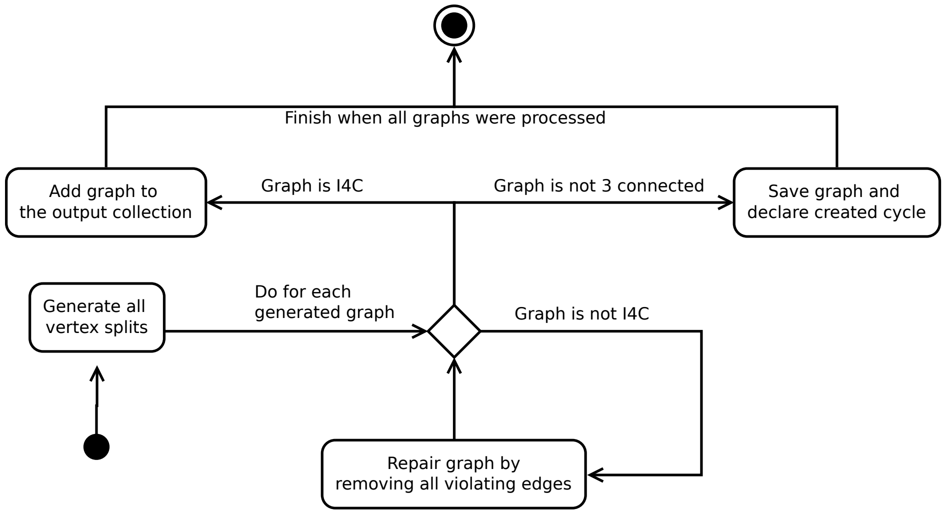

Figure 5.3: Scheme of generating and repairing vertex splits.

indicates its occurrence and continues processing the remaining graphs. Occurrence of a violating split means that in addition, Theorem 3.18 will be applied to G_0 in order to discover internally 4-connected graphs with both G_0 and G_1 through G_k minors.

Note that it is possible that after removing some violating edges from a graph G, it may no longer be 3-connected. However, it can still contain edges with endvertices having a common cubic neighbour. Such edges can no longer be called violating as the notion of a violating edge is defined for almost internally 4-connected graphs, which are required to be 3-connected. However, it is only natural to treat such edges as "violating"and continue removing until no such edges exist.

All generated internally 4-connected graphs are accompanied by canonical labeling (see Section 5.4) and organized to a set. Thus, only one representative of each isomorphism class gets to the output collection $\mathcal{A}$.

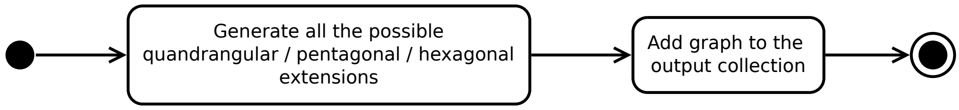

Figure 5.4: Scheme of generating quadrangular, pentagonal and hexagonal extensions.

5.3 Special extensions

Should quadrangular, pentagonal or hexagonal additions also be generated, the algorithm exhaustively explores all the combinations of vertices that meet the conditions defined by the operations. If a combination matching the criteria is found, the particular extension is produced. Note that in all the cases, this extension must be internally 4-connected and thus, it can be added to the output collection, which is free from isomorphisms due to the use of canonical representatives.

5.4 Representing the graphs and use of the canonical form

Our implementation of the splitter theorems represents graphs as data structures that carry information about the undirected graph in an adjacency matrix. To represent the matrix, we use a simple array of C/C++ `unsigned int` numeric records with the record at index i representing the adjacency vector for vertex i. This way, we gain all the benefits of the adjacency matrix representation (especially the constant time adjacency test and fast neighbourhood modification) while reducing the overhead of copying a 2-dimensional array. In addition to this fundamental data, we also maintain the following records for every graph H:

- The information about the origin of the graph. This includes the sequence of splitting operations leading to H from the preceding internally 4-connected graph G.

- The graph F that graph G was obtained from. This information can be used in connection with the former to quickly find a forbidden minor in H is some cases.

- The canonical representation of the graph H.

One of the principles making our search feasible is to never never compute anything twice for the same graph. Thus, every graph also carries its canonical representation, which we compute using nauty [23]. This representation has the form of a string — it is the adjacency matrix of the canonical representative of the isomorphism class read row-wise. The canonical form is used in order to maintain the collections $\mathcal{A}, \mathcal{P}$ and $\mathcal{M}$. Graphs in these collections are represented only using the canonical strings. The lexicographic ordering on the string also defines the ordering on graphs that is used for fast lookup in binary fashion.

The data structure used to represent graphs is one of the features that can be further optimized in the future. It is important to realize that the software is determined to handle a large number of graphs that need to be copied and passed around a lot. Thus, reducing the size of the representation will bring a performance improvement, which should be noticeable especially if the tools are used in a massively parallel environment.

5.5 Searching for minors

Considering the number of graphs produced by the splitting process, having an effective way of searching for forbidden minors turns out to be crucial. This part of our software was a bottleneck in both our Bachelor's research tools and the consequently created second generation. This section provides an in-depth description of the routine used for searching for minors in graph. Most of the functionality described was implemented by the second generation software, the heuristic presented at the end of the section was added in third generation.

Before we approach a detailed description of the heuristic used, let us first focus on searching for minors in graphs from outside of the box. Knowing that minor search is a lengthy procedure we should obviously attempt to avoid the search as much as possible. Thus, searching for minors in the generated graphs is always the very last operation that determines whether an internally 4-connected graph belongs to the graphs that possibly have a finite planar emulator. Also, if minor search is necessary, we have our tools remember the result for future use. The results are stored in the form of triples $(G, M, bool) \in \mathcal{M}$ where G is a graph, M is the tested minor, and *bool* is the result of the conducted search, i.e. true if $M \leq G$ and false otherwise. The graph G is stored via canonical labeling (see Section 5.4), M is a string identifier denoting the minor, and the last parameter a boolean value. Should the same search be initiated multiple times, the answer is first searched among the known results and the actual search routine is run only if the result is not found.

Now consider another aspect of the minor search process. Generating from an internally 4-connected graph G, by application of the splitter theorems, we obtain some other internally 4-connected graph H. If H does not contain any of the forbidden minors $M \in \varrho$, it is added to the possible planar-emulable graphs in $\mathcal{A}$ and the generating process is recursively applied to H. If J is a graph obtained from H by application of Theorems 3.18, 3.19 or 3.20, it was created by a series of small local changes in H. Such a nature of operations applied to graphs has the following effect: Let σ denote the sequence of operations that produces internally 4-connected graph $J^{\times} = \sigma(H)$ with a forbidden minor $M \in \varrho$ from H, due to which $J^{\times}$ is rejected. It is likely that σ is also applicable to graph J and produces an internally 4-connected graph $K = \sigma(J)$. In such a situation, we would be forced to search for minors from ϱ in K. Since we are already familiar with some structural relations among the graphs in question, we may observe that K also contains forbidden minor M.

Lemma 5.2. *Let H, J, K be non-projective internally 4-connected graphs with $H \leq J \leq K$. Denote σ the sequence of split operations such that $\sigma(J) = K$. If σ is applicable to H (it does not violate the conditions under which the operations of*

σ are applicable) and $\sigma(H)$ contains a M minor, then K also contains M minor.

Proof. We know that $H \leq J \leq K = \sigma(J)$. Since σ is applicable to both H and J, there is a subgraph S isomorphic to H in J, on which σ acts. Thus, $M \leq J^{\times} = \sigma(H) = \sigma(S) \leq \sigma(J) = K$. □

In our generating scheme, for an internally 4-connected graph J, we always have information about the preceding internally 4-connected graph H from which J was obtained. We also have the information about all internally 4-connected graphs $\mathcal{J}^{\times}$ and $\mathcal{J}^{\checkmark}$ (note that $J \in \mathcal{J}^{+}$) and some minors that they do or do not contain. Thus, when we need to verify that an internally 4-connected graph $K = \vartheta(J)$ obtained from J by sequence of operations ϑ does not contain any minor from ϱ, we take graph H, attempt to construct an internally 4-connected graph $\vartheta(H)$ and if successful (it is possible that ϑ does not apply to H), for each minor $M \in \rho$ we first verify that we did not already perform search $(\vartheta(H), M)$ with a positive answer. If there is $(\vartheta(H), M)$ with $M \leq \vartheta(H)$ known, we know that K contains M as well. Thus, adding $(K, M, true)$ to the list of known minor containment, we can reject K. Note that we cannot make any conclusion about M being a minor of K from the negative records $(\vartheta(H), M, false)$. Hence for all $M \in \varrho$, if there is no positive knowledge about $M \in \vartheta(H)$, we have no means of avoiding searching for forbidden minors in K, and the search routine has to be initiated.

Reusing the computed information about minors in graphs was first implemented by the second generation of our tools and further improved (in the sense of storing and accessing the records) in the third generation.

Having an internally 4-connected graph G, we need to generate all the internally 4-connected graphs H with G minor that do not contain any minor from the set of forbidden graphs ϱ. The set ρ is always formed by graphs that are known not to have finite planar emulators (2 disjoint k-graphs and $K_{3,5}$), graphs that are disregarded (usually the family of $K_7 - C_4$), and some additional internally 4-connected graphs that the generating scheme will be applied to separately. Let us denote the last group of graphs by $\mathcal{Q}$.

Consider two internally 4-connected non-projective graphs G_1, G_2 and their two sets of forbidden minors ϱ_1, ϱ_2 with $\mathcal{Q}_1, \mathcal{Q}_2$ such that G_2 is contained in ϱ_1 and it is the only graph in which ϱ_1, ϱ_2 differ. Formally, we have $\varrho_1 \setminus \varrho_2 = \{G_2\} = \mathcal{Q}_1 \setminus \mathcal{Q}_2 = \{G_2\}$, and we wish to generate all the internally 4-connected graphs with G_1 and G_2 minors.

One possible approach would be to start generating from G_1 using ϱ_1 yielding a set of internally 4-connected graphs with G_1 minors only — all the graphs with G_2 minor would be rejected as they contain minor $G_2 \in \varrho_1$. Then, the splitting process would be applied to G_2 using ϱ_2 producing a set of internally 4-connected graphs with both G_1 and G_2 minors, and a set of internally 4-connected graphs with only G_2 minors.

The other option is to run the search from G_2 first using ϱ_2. This produces a set of graphs that contains either both G_1 and G_2 minors, or G_2 minor only. Let this set be denoted by $\mathcal{A}_{\mathcal{G}_\in}$. As we know this set of results, we can employ the following optimization: Conduct the search from G_1 using ϱ_2. When graph H with $G_1 \leq H$ is generated, verify that H is not contained in $\mathcal{A}$. If it is, H contains G_2 minor and should be rejected.

The later approach avoids heuristic search for G_2 minor when generating from graph G_1. Instead, it performs a lookup in a per expectation small ordered set and thus, it can be significantly faster.

5.5.1 Heuristic for minor search

In our original work, we used an algorithm that exhaustively explored subgraphs which upon contraction could produce the minor being searched for. We conducted the computations in a highly parallel environment which made it feasible to produce the aforementioned results. The performance of this module, however, remained one of the limiting factors for finishing the computation for the graphs $K_{4,5} - 4K_2$ and $\mathcal{E}_2$.

Given the nature of the generated graphs, we can employ the following heuristic searching for minor minimal non-projective obstructions of our interest:

All the minors M that we need to search for in a graph G, produced by the splitting process, are non-projective graphs. Thus, if a contraction of a graph G', subgraph of G, results in a graph with projective embedding, it is not a contraction of our interest and it can be disregarded. Given the nature of our problem, all the graphs G produced by the splitting process are not far from being projective. Hence, we can expect many subgraphs G' of G to be forbidden for a contraction making the search for minor M quite straightforward.

The routine searching for non-projective minor M in a non-projective graph G starts with enumerating all the subgraphs of G of size at most $V(G) - V(M)$ vertices. Let us denote the set of such subgraphs $\mathcal{S}$. Then, for every $S \in \mathcal{S}$, we construct a graph $F_S \leq G$ by contracting all the vertices in S into one. Then, if F_S is not projective planar, S is excluded from $\mathcal{S}$ as its contraction cannot result in discovery of minor M in G.

In the rest of this section, we provide a brief description of the algorithm we use for testing existence of a projective embedding of a given graph $S \in \mathcal{S}$. For the algorithm to be usable in the aforementioned heuristic, it is required to perform very fast. First, let us remark that we are not aware of implementation of any projective plane embedding tests in any of the commonly use libraries that work with graphs[3]. Hence, our own implementation was necessary. There is an algorithm with running time in $\mathcal{O}(n)$ known, however its implementation is an extremely difficult task [31]. For this reason, we decided to to use an $\mathcal{O}(n^2)$ algorithm by W. Myrvold [24], implementation of which is claimed to be much easier [24, 25]. Given that graphs that this routine is applied to are rather small, the performance drop should not be as significant.

The projective test algorithm (Figure 1) starts with verifying if graph satisfies Euler's formula for the projective plane. If it tests whether the graph is planar. If graphs is planar, it is also projective planar and a plane embedding of G can be returned as result. Otherwise, a Kuratowski subgraph, i.e. subdivision K of $K_{3,3}$ or K_5, of graph G is found. We denote this graph by

3. The availability of such algorithms was discussed [25] with the author of [24]. The author offered to provide us with testing implementation of their algorithm, however we were warned that give its age, it no longer compiles on the currently used platforms. For this reason, we decided implement the algorithm on our own

K.

For each projective plane embedding of K, the main idea is to try to embed the rest of the graph in the faces of the embedding $\tilde{K}$ of K. A *bridge* of a graph G with respect to an embedded subgraph H is a subgraph of G which is either (1) an edge not in H whose endpoints are both in H plus its endpoints; or (2) a connected component $G - H$ together with the edges which connect a vertex in the connected component to a vertex in H and their endpoints. The vertices that a bridge shares with H are called its *attachment vertices*.

A bridge can be *drawn* in a face F if its all attachment vertices lie on F. A bridge B can be *embedded* in a face F is there is a planar embedding of $B \cup F$. A *k-face bridge* with respect to an embedded subgraph $\tilde{K}$ is a bridge that can be embedded in k faces. All bridges for an embedding $\tilde{K}$ of $K \in \{K_{3,3}, K_5\}$ in the projective plane are either 1-face, 2-face or 3-face bridges. Two bridges are *compatible for a face* if both can be drawn inside the face simultaneously. Otherwise, they are said to be conflicting.

The goal of the algorithm is to embed the bridges to faces, which can be perceived as an instance of a SAT problem — to determine if there is an assignment of boolean values to the variables of a formula in shape of conjunction of clauses, where every clause is formed by disjunction of literals. The SAT problem is generally known to be $\mathbb{NP}$-hard, however if every clause is formed precisely by two literals, so called 2-SAT problem, there is a simple algorithm running in linear time with respect to the input size.

The obstacle for SAT to 2-SAT reduction are the 3-face bridges. There are in total 3 ways in which K be embedded in the projective plane (cf. Figures 5.5 and 5.6), with 6 non-equivalent labellings for $K_{3,3}$ and in total 27 non-equivalent labellings for K_5. By case analysis of possible 3-face bridges with respect to a given type of embedding,[24] shows that for each of them, there is only a constant number of assignments of that bridge to the faces. Thus, for every such assignment of 3-bridges to faces, the problem can be reduced to an instance of 2-SAT and solved efficiently.

The running time $O(n^2)$ is obtained as follows: Let n denote the number of vertices and m be the number of edges of the input graph G. If there are

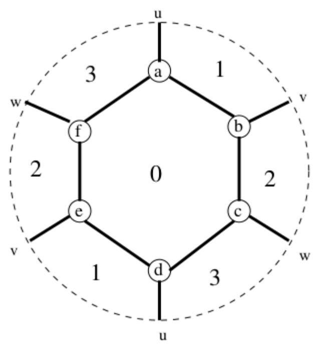

Figure 5.5: Projective embedding of $K_{3,3}$.

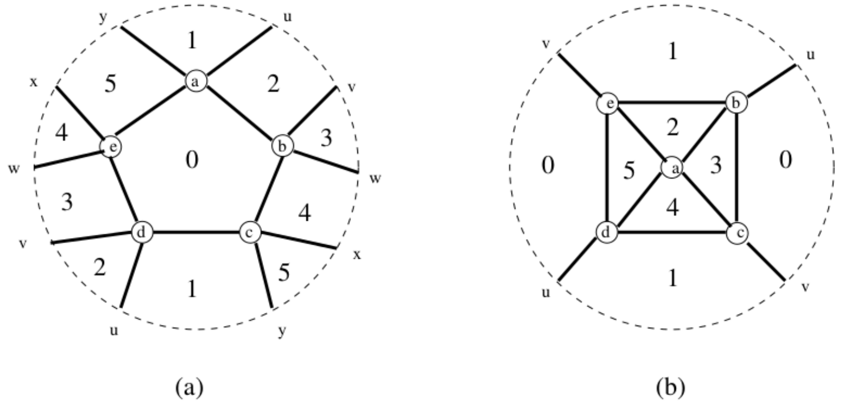

Figure 5.6: Two ways of embedding K_5 in the projective plane.

more than $3n-3$ edges, the input graph cannot be projective by Euler's formula for the projective plane (see Theorem 2.10) and the algorithm rejects in constant time after reading the input. Thus for the remaining steps, we have at most $3n-3 \in O(n)$ edges in the graph.

If G is planar, is is also projective planar, which is detected by line (4). This test can be conducted in $O(n)$ time using any planarity testing algorithm, for example [3], which can also extract the Kuratowski subgraph K if G is non-planar. For both the aforementioned tasks, our implementation uses the aforementioned Boyer-Myrvold planarity test and Kuratowski obstruction extraction algorithm [3] available in C library boost [29]. An independent implementation is also available in OGFD [5].

The loop at line (8) is executed at most $O(1)$ times — precisely 6 times for K homeomorphic to $K_{3,3}$ and 27 times for K homeomorphic to K_5.

For each of the embeddings $\tilde{K}$, the algorithm finds the bridges of $\tilde{K}$ and faces in which the bridges can be embedded using a modified BFS in $O(n)$ time. Determining the faces that each bridge can be embedded in can be accomplished using a planarity test. The total number for all tests is in $O(n)$ using a linear time planarity tester if the graph tested for planarity is the bridge with its points of attachment connected in a cycle which respects the cyclic order of the face. The number of faces in $\tilde{K}$ is constant, thus line (9) operates in $O(n)$ time.

The conflicts among bridges can be computed using a fairly small finite state machine (10 states, each of them with out-degree 3) described by [24]. It is important to realize that the number of bridges can be linear and thus the number of bridge pairs can be quadratic. In order for the entire algorithm to run in $O(n^2)$ time, the information about bridge conflicts must be computed in $O(n^2)$ total time and thus, this part of the algorithm (which also utilizes $O(n)$ number of edges) really is crucial.

For loop at line (14), the number of arrangements of 3-face bridges is constant as state earlier. The search for an assignment of bridges to faces is conducted via reduction to 2-SAT instance which is then solved in a linear time, thus the total run time is in $O(n^2)$. If the projective embedding exists, it can be composed using the information available in $O(n)$ time and returned.

Because each of the loops involves in the above code only involves a constant number of iterations, the time for the whole algorithm is in $O(n^2)$. The author also remarks that the constant overhead of this algorithm is very reasonable.

Algorithm 1 Algorithm searching for projective embedding of a graph G.

if $m > 3n - 3$ **then**
    return false.
end if
if G is planar **then**
    return planar embedding of G
end if
Find subgraph K homeomorphic to $K_{3,3}$ or K_5.
for all labeled projective plane embedding $\tilde{K}$ of K **do**
    Find all the bridges of $\tilde{K}$ and determine which faces they can be embedded in.
    if bridge b cannot be embedded in any face of $\tilde{K}$ **then**
        return false.
    end if
    Compute the conflicts between pairs of bridges.
    for all arrangements of 3-face bridges do **do**
        Use reduction to 2-SAT problem instance in order to search for a compatible bridge assignment to faces.
        if there is such an assignment of bridges to the faces of $\tilde{K}$ **then**
            return a projective planar embedding
        end if
    end for
end for
return false.

Chapter 6

Generated results

In Chapter 3, Section 3.1, we presented two tools that can be used for an exhaustive search for non-projective internally 4-connected graphs. We used both Theorem 3.18 and combination of Theorems 3.19 and 3.20 for a computerized exhaustive search for the internally 4-connected graphs that contain one of the $K_{4,4}-e, K_{4,5}-4K_2, K_{1,2,2,2}, \mathcal{B}_7, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2$ or $\mathcal{E}_2$ minors, can have a finite planar emulator and do not contain any minor isomorphic to a graph in the $K_7 - C_4$ family, i.e. we searched for graphs without a minor isomorphic to a graph in the set $\Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,4}, K_7\}$.

The search solely via Theorem 3.18 was described in [6]. This section presents the search and results obtained by application of Theorems 3.19 and 3.20. The theorems were applied in the following fashion: If the initial internally 4-connected graph G satisfies condition specified by Theorem 3.20, this theorem is applied and only t-step edge additions are generated from the graph H that is a t-step addition of G, $t \geq 2$. If the conditions are not satisfied, Theorem 3.19 is applied and all of t-step edge addition, vertex splits, quadrangular, pentagonal an hexagonal extensions are generated. If application of Theorem 3.19 produces after a vertex split graph H that is not internally 4-connected, all the violating edges and consequently edges that connect neighbours of a cubic vertex are removed to obtain graph $F \leq H$. If F is internally 4-connected, it is used as an input graph for further generating using Theorem 3.19. If the graph is not internally 4-connected (in fact, this can happen only if F is not cubic), we call F a *violating minor* as it does not allow for further application of any of the theorems. The vertex split that produced H from G is referred to as a *violating split*. Note that these situations are common e.g. for graph $\mathcal{E}_2$ and we provide a detailed analysis later. In such a case, we apply also Theorem 3.18 to graph G in order to check whether some other internally 4-connected graphs $H' \geq G$ exist.

In the following sections, we list the generated results. Together with each graph, we list also a reference to this result in [6], where it was generated using Theorem 3.18. In all the cases, we refer to labeling as in Figure 4.1. The results are listed in separate sections based on from which internally 4-connected minor minimal obstruction for the projective plane they were generated. For every such minor minimal obstruction, we also include a diagram of the generating process (Figures 6.1 through 6.7). These figure contain the splitting tree where every node represents a graph explored within our search. The color coding has the following meaning.

- The blue nodes represent internally 4-connected graphs for which we do not know if a finite planar emulator exists.
- The yellow nodes are the graphs in the family of $K_{1,2,2,2}$.
- The green nodes are the graphs for which a finite planar emulator can be derived from the previously discovered ones. Some of these graphs — the ones that do not contains an $\mathcal{E}_2$ minor, were already described by Hliněný in [20].
- The white nodes represent the graphs that were created for technical reasons only so that Theorem 3.18 can be applied to them. They are always isomorphic to their predecessor which has a violating minor (see Section 6.8).
- Specifically in the generating tree for $\mathcal{E}_2$, the colour coding has a slightly different meaning. The green graphs are the graphs that belong to class $\mathcal{IE}_2^{[0]}$, the orange graphs are the graphs from $\mathcal{IE}_2^{[2]}$ and the blue graph is the only graph from $\mathcal{IE}_2^{[4]}$ (see Section 6.8 for more details). The white nodes have the meaning stated above.

The directed edges in figures are labelled by sequences of splitting operations. The edge addition is denoted by `EA`, the edge removal by `ER`, the quadrangular, pentagonal and hexagonal extension by `QE`, `PE` and `HE` with standard parameters respectively, and the vertex split by `VS`. The parameters for vertex split denote the vertex v to be split and the bit mask of the neighbourhood $N(v)$. The mask neighbours will become the neighbours of the vertex introduced by the split, the unmasked neighbours will remain adjacent to v. The vertex introduced by the vertex split gets the next available label $l \in \mathbb{N}$.

6.1 $K_{4,4} - e$

Consider the graph $K_{4,4}-e$. Let the set ϱ be the set of graphs $\{K_{1,2,2,2}$, $\mathcal{B}_7$, $\mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2, K_{4,5} - 4K_2\}$ and let G be an internally 4-connected graph obtained from $K_{4,4}-e$ by Theorem 3.19 that contains a minor M isomorphic to a member of ϱ. By Theorem 3.19, there exist a generating sequence leading to G from M. Hence, we do not need to consider G here, because it will be obtained later from M. The set of forbidden minors for $K_{4,4}-e$ thus is

$$\varrho_0 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \Delta\mathrm{Y}(K_{1,2,2,2}) \cup \{K_{4,4}, K_7\} \cup \{K_{4,5} - 4K_2\}$$

(see Corollary 3.12 for the rest of forbidden minors). The 85 internally 4-connected graphs that can be obtained from $K_{4,4} - e$ and do not contain a minor isomorphic to a member of ϱ_0 are listed in Appendix A.3. We denote this set of results by $\mathcal{A}_{K_{4,4}-e} = K_{4,4}-e^{[i]}$, $0 \leq 72$. The tree depicting the generating process is shown in Figure 6.1.

6.2 $K_{1,2,2,2}$

For the graph $K_{1,2,2,2}$, the set of forbidden minors is

$$\varrho_1 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,4}, K_7\} \cup \{\mathcal{B}_7, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2\} \cup \{K_{4,5} - 4K_2\}.$$

The same arguments as above apply for inclusion of the graphs $\mathcal{B}_7$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{D}_2$ and $\mathcal{E}_2$. Also, note that graph $K_{4,4} - e$ cannot be in ϱ_1. The internally 4-connected graphs that can be obtained from $K_{1,2,2,2}$ and do not contain a minor isomorphic to a member of ϱ_1 are as follows:

- $K_{1,2,2,2}^{[0]} = K_{1,2,2,2}$
- $K_{1,2,2,2}^{[1]} = K_{1,2,2,2}^{[0]} {<} 6\left\{\begin{smallmatrix}0,2,3\\1,4,5\end{smallmatrix}\right\}$, in [6] referred to as $K_{1,2,2,2}^{[1]}$

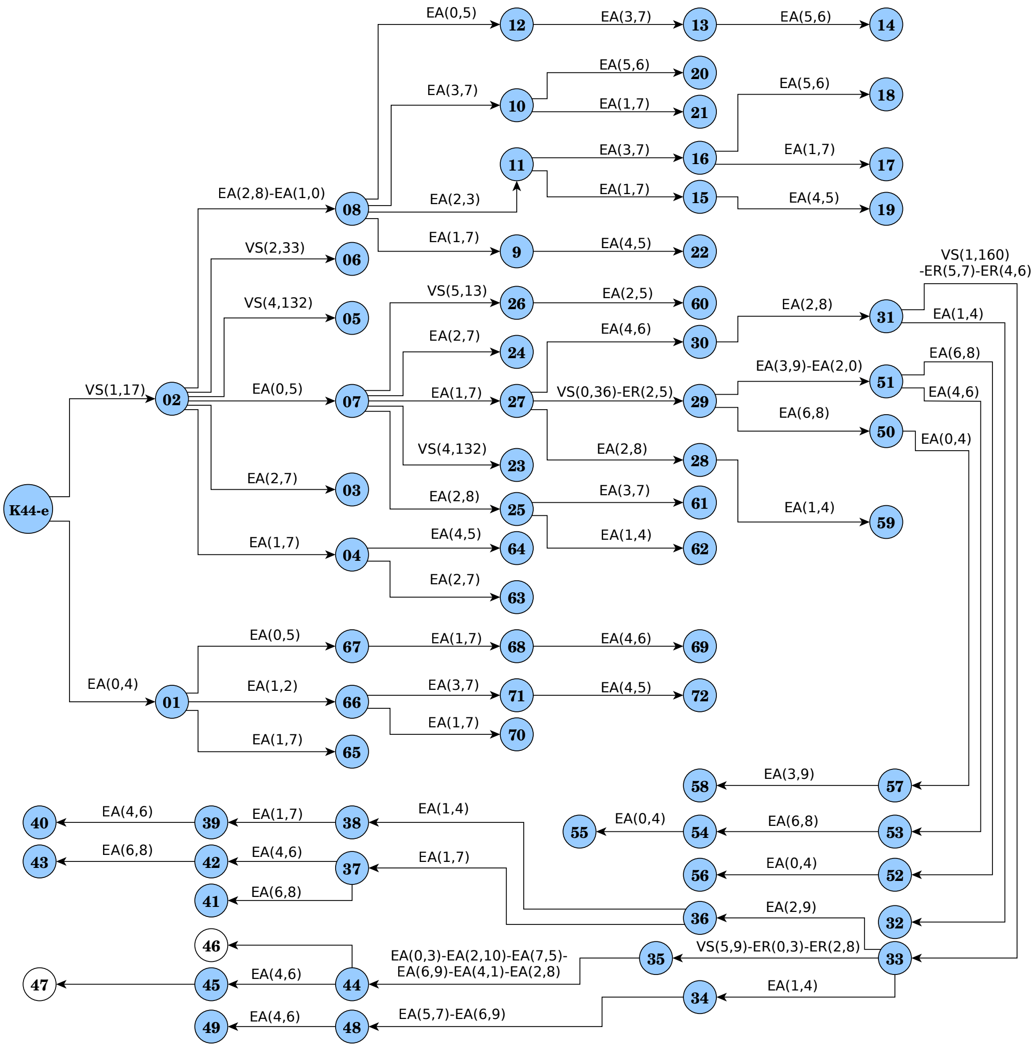

Figure 6.1: A tree depicting the generating process for $K_{4,4}-e$.

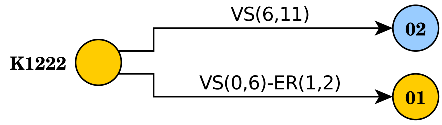

Figure 6.2: A tree depicting the generating process for $K_{1,2,2,2}$. The represented by the orange belong to the family of $K_{1,2,2,2}$.

There are no other graphs that can be created from $K_{1,2,2,2}$ by application of Theorem 3.19 (note that Theorem 3.20 is not applicable here). All the 1-step addition extensions are isomorphic to $K_{1,2,2,2}+\{0,3\}$, which contains a $K_7 - C_4$ minor. Hence, no t-step addition extensions with $t \geq 1$ comes into account either. There are no quadrangular, pentagonal or hexagonal extensions that could be obtained from $K_{1,2,2,2}$. The only remaining operations to explore are vertex splits, where without loss of generality:

- $K^{[0]}_{1,2,2,2} < 6\left\{\substack{0,4,5\\1,2,3,}\right\}$ contains $\mathcal{D}_{17} \in \Lambda$ minor
- $K^{[0]}_{1,2,2,2} < 6\left\{\substack{2,5\\0,1,3,4,}\right\}$ contains $K_{3,5} \in \Lambda$ minor
- $K^{[0]}_{1,2,2,2} < 5\left\{\substack{1,4,6\\0,3}\right\}$ contains $\mathcal{D}_3 \in \Delta\mathrm{Y}(K_7 - C_4)$ minor
- $K^{[0]}_{1,2,2,2} < 0\left\{\substack{1,2\\4,5,6}\right\}$ is not internally 4-connected, but via removing edge $\{1,2\}$, it contains an internally 4 subgraph with $\mathcal{B}_7 \in \varrho_1$ minor
- $K_{1,2,2,2} < 1\left\{\substack{2,3,5\\0,6}\right\}$ is not internally 4-connected, but via removing edge $\{0,6\}$, it contains an internally 4 subgraph with $\mathcal{D}_3 \in \Delta\mathrm{Y}(K_7 - C_4)$ minor

All the other operations create a graph isomorphic to one of the mentioned above.

When applying Theorem 3.19 on the graph $K^{[1]}_{1,2,2,2}$, all the t-step addition extensions can be transformed to t-step addition extensions of $K^{[0]}_{1,2,2,2}$ by the backward contraction of the edge $\{6,7\}$ (resulting in possible parallel edges in the cases of $K^{[1]}_{1,2,2,2}+\{3,6\}$ and $K^{[1]}_{1,2,2,2}+\{5,7\}$) and as such, they contain the corresponding minors listed above (up to symmetries). There are no quadrangular, pentagonal or hexagonal extensions of $K^{[1]}_{1,2,2,2}$. Any graph G resulting from $K^{[1]}_{1,2,2,2}$ by splitting a vertex contains a minor isomorphic to the graph H resulting from $K^{[0]}_{1,2,2,2}$ by performing the corresponding vertex split via contraction of the edge $\{6, 7\}$ (respectively $\{8, x\}$ with $x \in \{6, 7\}$ if the edge $\{6, 7\}$ does not exist in some cases of splitting of the vertex 6 or 7) in G. Note that H must be different from $K^{[1]}_{1,2,2,2}$, and therefore it, as well as G, contains a minor isomorphic to one of the graphs, members ofϱ_1, mentioned above.

Hence, the graphs $K^{[0]}_{1,2,2,2}$ and $K^{[1]}_{1,2,2,2}$ are the only results. Let us denote them by $\mathcal{A}_{K_{1,2,2,2}} = \{K^{[0]}_{1,2,2,2}, K^{[1]}_{1,2,2,2}\}$.

6.3 $\mathcal{B}_7$

The set of forbidden minors for the graph $\mathcal{B}_7$ is

$$\varrho_2 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,4}, K_7\} \cup \{\mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2\} \cup \{K_{4,5} - 4K_2\}.$$

The same arguments as above apply for inclusion of the graphs $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{D}_2$, $\mathcal{E}_2$. Also, note that the graphs $K_{4,4}-e$ and $K_{1,2,2,2}$ cannot be in ϱ_2. The internally 4-connected graphs that can be obtained from $\mathcal{B}_7$ and do not contain a minor isomorphic to a member of ϱ_2 are listed below.

Let us start generating graphs from $\mathcal{B}_7$ (via Theorem 3.19, Theorem 3.20 is not applicable here). Only the following graphs without a minor isomorphic to a member of ϱ_2 can be created:

- $\mathcal{B}^{[0]}_7 = \mathcal{B}_7$
- $\mathcal{B}^{[1]}_7 = \mathcal{B}^{[0]}_7 + \{6, 7\}$, in [6] referred to as $\mathcal{B}^{[2]}_7$
- $\mathcal{B}^{[2]}_7 = \mathcal{B}^{[0]}_7 < 1\left\{{0,2 \atop 3,5,6}\right\}$, in [6] referred to as $\mathcal{B}^{[3]}_7$
- $\mathcal{B}^{[3]}_7 = \mathcal{B}^{[0]}_7 < 0\left\{{1,5 \atop 6,7}\right\} - \{1, 5\} \simeq \mathcal{C}_3$ (the vertex split itself does not create an internally 4-connected graph)
- $\mathcal{B}^{[4]}_7 = \mathcal{B}^{[0]}_7 < 6\left\{{0,1,3 \atop 2,4,5}\right\}$, in [6] referred to as $\mathcal{B}^{[4]}_7$
- $\mathcal{B}^{[5]}_7 = \mathcal{B}^{[0]}_7 < 1\left\{{3,5 \atop 0,2,6}\right\} - \{3, 5\} \simeq \mathcal{C}_4$ (the vertex split itself does not create an internally 4-connected graph)

As for the other operations, there are three more pairwise non-symmetric options of adding an edge, which are $e_1 = \{1,4\}$, $e_2 = \{5,7\}$ and $e_3 = \{0,2\}$. The graphs $\mathcal{B}_7^{[0]} + e_1$ and $\mathcal{B}_7^{[0]} + e_2$ both contain a minor isomorphic to $K_7 - C_4$. The edge e_3 in $\mathcal{B}_7^{[0]} + e_3$ is violating and cannot be "repaired" in one step. Hence, no addition extensions of $\mathcal{B}_7^{[0]}$ other than $\mathcal{B}_7^{[1]}$ are possible.

No quadrangular, pentagonal or hexagonal extensions of $\mathcal{B}_7^{[0]}$ exist.

The rest of possible pairwise non-isomorphic graphs that can be created by splitting a vertex is covered by the following:

- $\mathcal{B}_7^{[0]} < 6\left\{{0,4 \atop 1,2,3,5}\right\}$, $\mathcal{B}_7^{[0]} < 6\left\{{0,2,4 \atop 1,3,5}\right\}$, and $\mathcal{B}_7^{[0]} < 6\left\{{0,4,5 \atop 1,2,3}\right\}$ contain a minor isomorphic to $\mathcal{E}_{20} \in \Lambda$
- $\mathcal{B}_7^{[0]} < 6\left\{{0,3 \atop 1,2,4,5}\right\}$ contains $K_{3,5} \in \Lambda$ minor
- $\mathcal{B}_7^{[0]} < 1\left\{{2,5 \atop 0,3,6}\right\}$ and $\mathcal{B}_7^{[0]} < 6\left\{{0,1,2,4 \atop 3,5}\right\} - \{3,5\}$ contains $\mathcal{F}_1 \in \Delta\mathrm{Y}(K_7 - C_4)$ minor. Note that the graph $\mathcal{B}_7^{[0]} < 6\left\{{0,1,2,4 \atop 3,5}\right\}$ is not internally 4-connected.
- $\mathcal{B}_7^{[0]} < 6\left\{{0,1 \atop 2,3,4,5}\right\} - \{0,1\}$ (the vertex split itself does not create an internally 4-connected graph) contains $\mathcal{D}_3 \in \Delta\mathrm{Y}(K_7 - C_4)$ minor.

By consequent applications of Theorem 3.19 (Theorem 3.20 is still not applicable), we obtain the following graphs that do not contain a minor isomorphic to a member of ϱ_2 (we no more include the rest of graphs that do not satisfy our requirements):

- $\mathcal{B}_7^{[6]} = \mathcal{B}_7^{[1]} < 6\left\{{0,1,3 \atop 2,4,5,7}\right\}$, in [6] referred to as $\mathcal{B}_7^{[7]}$
- $\mathcal{B}_7^{[7]} = \mathcal{B}_7^{[1]} + \{0,2\}$, in [6] referred to as $\mathcal{B}_7^{[1]}$
- $\mathcal{B}_7^{[8]} = \mathcal{B}_7^{[1]} < 6\left\{{0,2,3 \atop 1,4,5,7}\right\}$, in [6] referred to as $\mathcal{B}_7^{[8]}$
- $\mathcal{B}_7^{[9]} = \mathcal{B}_7^{[2]} < 6\left\{{0,1,4 \atop 2,3,5}\right\}$, in [6] referred to as $\mathcal{B}_7^{[10]}$
- $\mathcal{B}_7^{[10]} = \mathcal{B}_7^{[6]} + \{0,2\}$, in [6] referred to as $\mathcal{B}_7^{[5]}$
- $\mathcal{B}_7^{[11]} = \mathcal{B}_7^{[6]} + \{0,4\}$, in [6] referred to as $\mathcal{B}_7^{[6]}$

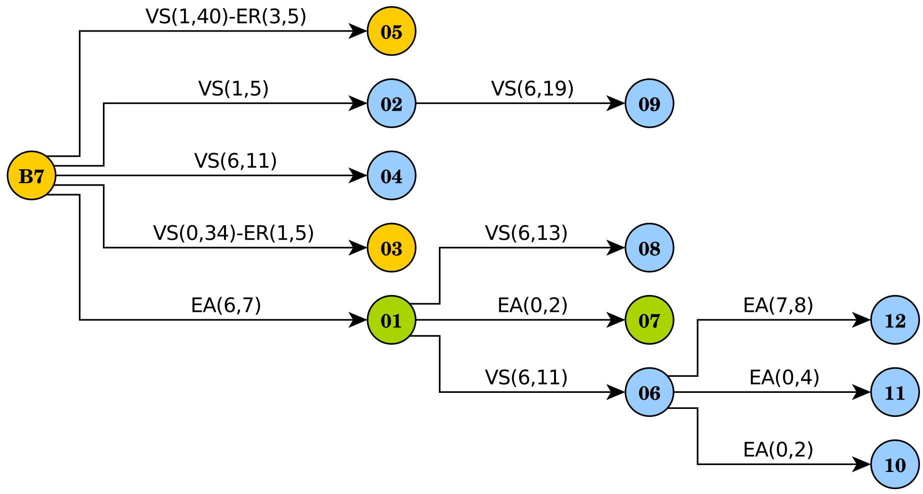

Figure 6.3: A tree depicting the splitting process for the graph $\mathcal{B}_7$. The orange highlighted graphs belong to the family of $K_{1,2,2,2}$. For the graphs in green, we have a finite planar emulator. Those graphs were also identified by Hliněný with respect to the finite planar covers problem [18, 20].

- $\mathcal{B}_7^{[12]} = \mathcal{B}_7^{[6]} + \{7, 8\}$, in [6] referred to as $\mathcal{B}_7^{[9]}$

Let us denote the set of results generated from the graph $\mathcal{B}_7$, excluding $\mathcal{B}_7^{[3]} \simeq \mathcal{C}_3$ and $\mathcal{B}_7^{[5]} \simeq \mathcal{C}_4$, by $\mathcal{A}_{\mathcal{B}_7}$. The cardinality of $\mathcal{A}_{\mathcal{B}_7}$ is 11.

6.4 $\mathcal{C}_4$

The set of forbidden minors for the graph $\mathcal{C}_4$ is $\varrho_3 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,4}, K_7\} \cup \{\mathcal{D}_2, \mathcal{E}_2\}$. The same arguments as above apply for inclusion of the graphs $\mathcal{D}_2, \mathcal{E}_2$. Also, note that the graphs $K_{4,4}-e$, $K_{1,2,2,2}$ and $\mathcal{B}_7$ cannot be in ϱ_3. The eight internally 4-connected graphs that can be obtained from $\mathcal{C}_4$ and do not contain a minor isomorphic to a member of ϱ_3 are as follows:

- $\mathcal{C}_4^{[0]} = \mathcal{C}_4$
- $\mathcal{C}_4^{[1]} = \mathcal{C}_4^{[0]} + \{0, 2\}$, in [6] referred to as $\mathcal{C}_4^{[1]}$

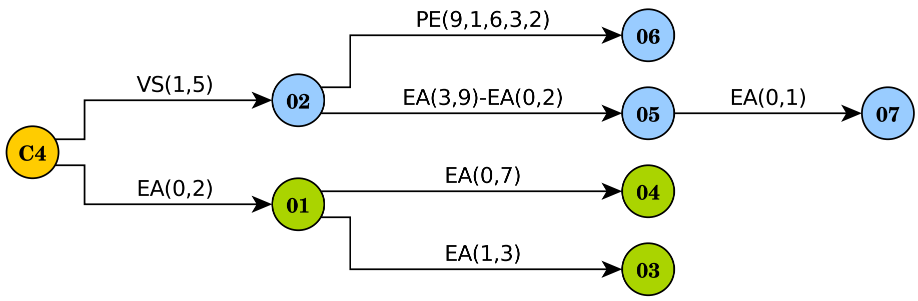

Figure 6.4: A tree depicting the generating process for $\mathcal{C}_4$.

- $\mathcal{C}_4^{[2]} = \mathcal{C}_4^{[0]} < 1\left\{{0,2 \atop 6,8}\right\}$, in [6] referred to as $\mathcal{C}_4^{[5]}$
- $\mathcal{C}_4^{[3]} = \mathcal{C}_4^{[1]} + \{1,3\}$, in [6] referred to as $\mathcal{C}_4^{[2]}$
- $\mathcal{C}_4^{[4]} = \mathcal{C}_4^{[1]} + \{0,7\}$, in [6] referred to as $\mathcal{C}_4^{[6]}$
- $\mathcal{C}_4^{[5]} = \mathcal{C}_4^{[2]} + \{3,9\} + \{0,2\}$, in [6] referred to as $\mathcal{C}_4^{[3]}$
- $\mathcal{C}_4^{[6]} = \mathcal{C}_4^{[2]} \pentagon \{9,1,6,3,2\}$, in [6] referred to as $\mathcal{C}_4^{[4]}$
- $\mathcal{C}_4^{[7]} = \mathcal{C}_4^{[5]} + \{0,1\}$, in [6] referred to as $\mathcal{C}_4^{[7]}$

Note that in this case, the graph $\mathcal{C}_4^{[5]}$ is obtained from $\mathcal{C}_4^{[2]}$ by 2-step addition extension. Since all the components of $\mathcal{C}_4^{[2]}$ induced by cubic vertices are trees or cycles, Theorem 3.20 is applicable and only t-step addition extensions of $\mathcal{C}_4^{[5]}$ need to be considered further (resulting in no graphs satisfying our criteria). Let the set of graphs generated from $\mathcal{C}_4$ be denoted by $\mathcal{A}_{\mathcal{C}_4}$, $|\mathcal{A}_{\mathcal{C}_4}| = 8$.

6.5 $\mathcal{C}_3, \mathcal{D}_2$

For the graphs $\mathcal{C}_3$ and $\mathcal{D}_2$, we list the 30 and 38 (respectively) generated graphs in the appendix.

The sets of forbidden minors are

$$\varrho_4 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,4}, K_7\} \cup \{\mathcal{C}_3, \mathcal{D}_2, \mathcal{E}_2\} \cup \{K_{4,5} - 4K_2\}$$
$$\varrho_5 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,4}, K_7\} \cup \{\mathcal{E}_2\} \cup \{K_{4,5} - 4K_2\}$$

for $\mathcal{C}_3$ and $\mathcal{D}_2$ respectively. See the previous sections for an explanation of how these sets were obtained.

Let us denote the set of the graphs obtained from $\mathcal{C}_3$ and $\mathcal{D}_2$ denote by $\mathcal{A}_{\mathcal{C}_3} = \{\mathcal{C}_3^{[m]}\}$ with $0 \leq m \leq 30$, $m \neq 5$, $\mathcal{A}_{\mathcal{D}_2} = \{\mathcal{D}_2^{[n]}\}$ with $0 \leq n \leq 39$, $n \neq 4$.

6.6 $\mathcal{E}_2$

One of the main goals of this work was to finish the computations for the graph $\mathcal{E}_2$ and provide a full list of internally 4-connected graphs that comply with the previously introduced notion and contain a graph isomorphic to $\mathcal{E}_2$ as a minor. Having heavily optimized our tools, especially by employing heuristic for the minor search (cf. Section 5.5.1), we were able to finish all the necessary computation for $\mathcal{E}_2$ and obtained results described in this section. We remark that these results were not published in [7].

The set of forbidden minors used for the graph $\mathcal{E}_2$ was

$$\varrho_7 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \{K_{4,5} - 4K_2\}.$$

As the cubic vertices in $\mathcal{E}_2$ do not induce a tree or a cycle, only Theorem 3.19 can be applied and vertex splits must be considered in any case. This holds also for some other internally 4-connected graphs that can be obtained from $\mathcal{E}_2$. In many (in fact all but three) cases, there are some vertex splits that produce a graph that is not internally 4-connected. Denote the initial graph by G and the graph obtained by a vertex split H. Recall that in such a situation, we iteratively remove the violating edges and consequently also the edges in the neighbourhood of a cubic vertex from H. If the resulting graph is not 3-connected, we cannot be sure that there is no other internally 4-connected graph J that contain both $G \leq H$ minors. Hence in addition to Theorem 3.19, we also apply Theorem 3.18 to G in

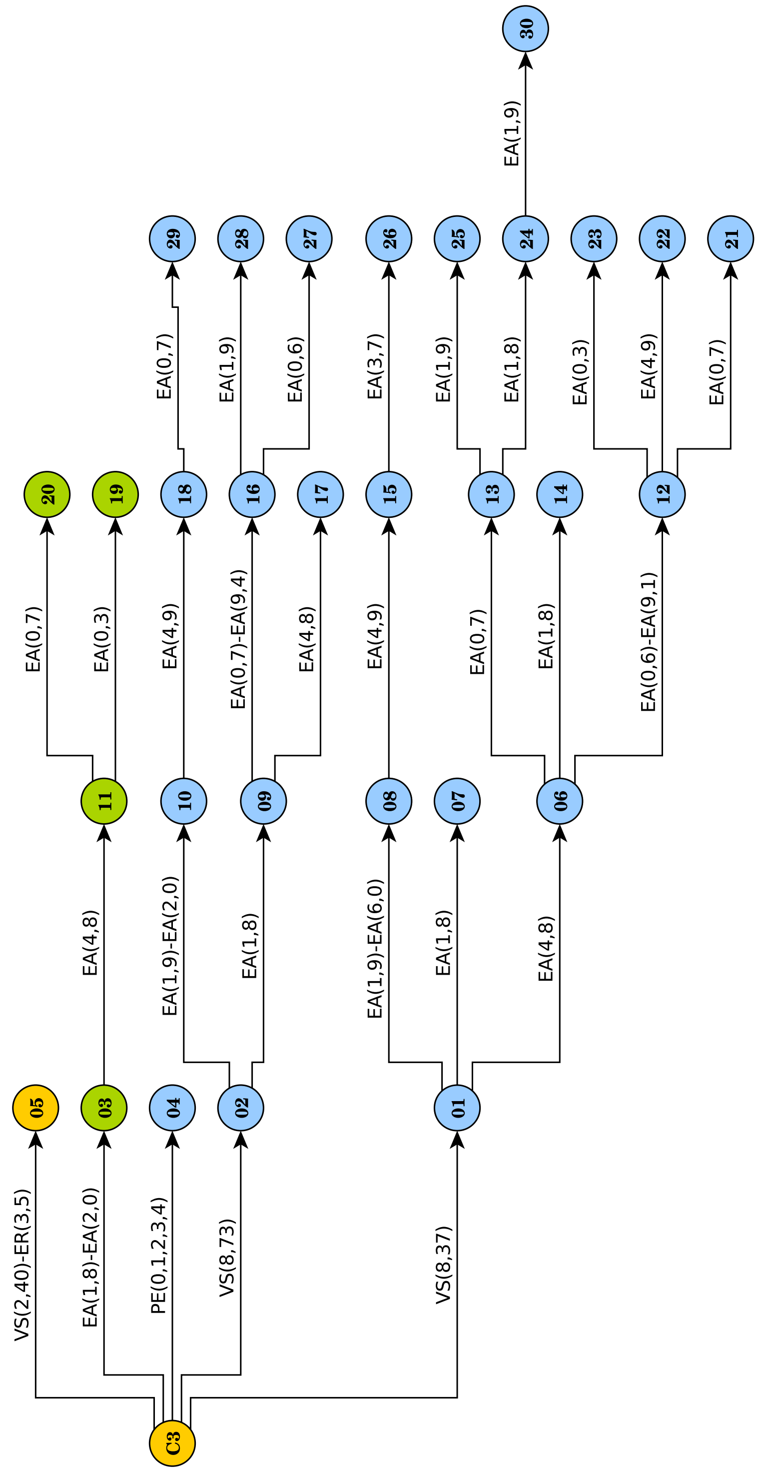

Figure 6.5: Generating tree for the graph C_3.

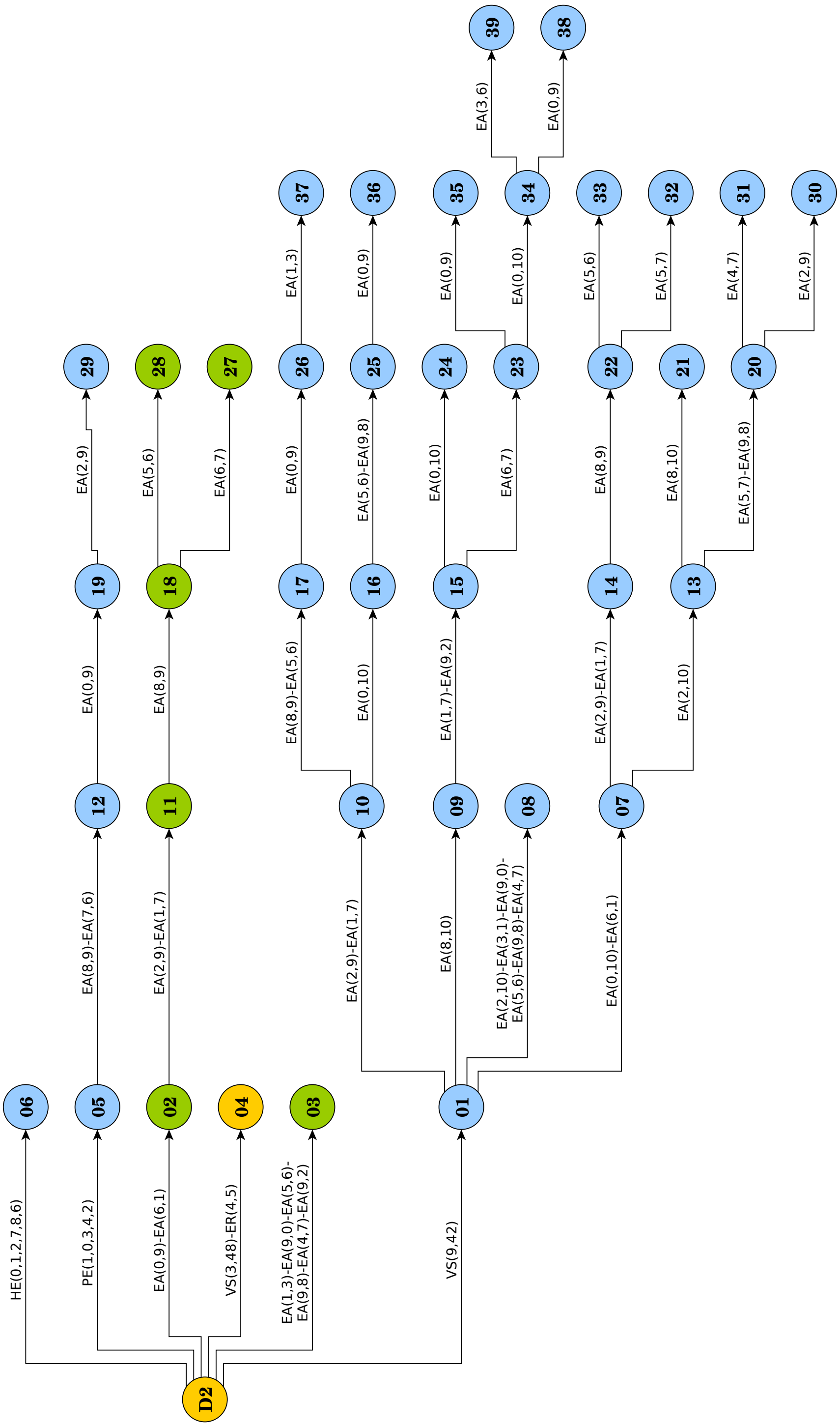

Figure 6.6: Generating tree for the graph $\mathcal{D}_2$.

order to generate J. For technical reasons, our tools perform this task by taking the initial graph $G \simeq \mathcal{E}_2^{[i]}$ for some $0 \leq i \leq 38$, assigning the next available index $i < j \leq 38$ to it effectively creating graph $\mathcal{E}_2^{[j]} \simeq \mathcal{E}_2^{[i]} \simeq G$ and applying Theorem 3.18 to $\mathcal{E}_2^{[j]}$. We list the results in the same form as we wish to keep the list consistent with the records in our logs. To clearly indicate that graph $\mathcal{E}_2^{[j]}$ was created for technical reasons only, we mark those cases with ✗. Note that all the graphs obtained from such a graph $\mathcal{E}_2^{[j]}$ (additionally marked with ✗) were obtained using Theorem 3.18. The total number of graphs generated from $\mathcal{E}_2$ is 21 and together, they form set denoted by $\mathcal{A}_{\mathcal{E}_2} = \{\mathcal{E}_2^{[p]}\}$, $0 \leq p \leq 38$.

- $\mathcal{E}_2^{[0]} = \mathcal{E}_2$
- $\mathcal{E}_2^{[1]} = \mathcal{E}_{2\ +\{2,9\}+\{10,8\}+\{5,7\}+\{10,6\}}^{[0]+\{1,2\}+\{10,0\}+\{5,3\}+\{10,4\}}$
- $\mathcal{E}_2^{[2]} = \mathcal{E}_2^{[0]} < 10\left\{\substack{1,2,5\\3,7,9}\right\}$
- $\mathcal{E}_2^{[3]} = \mathcal{E}_{2\ +\{10,4\}+\{7,9\}+\{10,8\}}^{[0]+\{1,2\}+\{10,0\}+\{5,3\}}$
- $\mathcal{E}_2^{[4]} = \mathcal{E}_2^{[0]} \hexagon \{0, 6, 4\}$
- $\mathcal{E}_2^{[5]} = \mathcal{E}_2^{[1]} < 10\left\{\substack{0,2,3,5\\1,4,6,7,8,9}\right\}$
- $\mathcal{E}_2^{[6]} = \mathcal{E}_2^{[1]} < 10\left\{\substack{0,2,3,7,8\\1,4,5,6,9}\right\}$
- $\mathcal{E}_2^{[7]} = \mathcal{E}_2^{[1]}$ ✗
- $\mathcal{E}_2^{[8]} = \mathcal{E}_2^{[7]} < 2\left\{\substack{0,1\\8,9,10}\right\} + \{10, 11\}$
- $\mathcal{E}_2^{[9]} = \mathcal{E}_{2\ +\{7,9\}+\{11,8\}+\{1,2\}}^{[2]+\{0,10\}+\{5,3\}+\{10,4\}}$
- $\mathcal{E}_2^{[10]} = \mathcal{E}_2^{[3]} + \{6, 10\}$
- $\mathcal{E}_2^{[11]} = \mathcal{E}_2^{[3]}$ ✗
- $\mathcal{E}_2^{[12]} = \mathcal{E}_2^{[11]} + \{1, 5\} + \{6, 10\}$
- $\mathcal{E}_2^{[13]} = \mathcal{E}_2^{[5]}$ ✗
- $\mathcal{E}_2^{[14]} = \mathcal{E}_2^{[17]} < 2\left\{\substack{8,9\\0,1,11}\right\} + \{11, 12\}$
- $\mathcal{E}_2^{[15]} = \mathcal{E}_2^{[6]}$ ✗
- $\mathcal{E}_2^{[16]} = \mathcal{E}_2^{[15]} < 2\left\{\substack{0,1\\8,9,11}\right\} + \{11, 12\}$
- $\mathcal{E}_2^{[17]} = \mathcal{E}_2^{[8]}$ ✗
- $\mathcal{E}_2^{[18]} = \mathcal{E}_2^{[17]} < 5\left\{\substack{3,4\\6,7,10}\right\} + \{10, 12\}$
- $\mathcal{E}_2^{[19]} = \mathcal{E}_2^{[9]} + \{6, 10\}$
- $\mathcal{E}_2^{[20]} = \mathcal{E}_2^{[9]} + \{6, 11\}$
- $\mathcal{E}_2^{[21]} = \mathcal{E}_2^{[9]}$ ✗
- $\mathcal{E}_2^{[22]} = \mathcal{E}_2^{[21]} + \{5, 7\} + \{6, 11\}$
- $\mathcal{E}_2^{[23]} = \mathcal{E}_2^{[21]} + \{1, 7\} + \{6, 11\}$
- $\mathcal{E}_2^{[24]} = \mathcal{E}_2^{[21]} + \{1, 5\} + \{6, 10\}$
- $\mathcal{E}_2^{[25]} = \mathcal{E}_2^{[10]}$ ✗
- $\mathcal{E}_2^{[26]} = \mathcal{E}_2^{[12]}$ ✗
- $\mathcal{E}_2^{[27]} = \mathcal{E}_2^{[14]}$ ✗
- $\mathcal{E}_2^{[28]} = \mathcal{E}_2^{[27]} < 5\left\{\substack{3,4\\6,7,11}\right\} + \{11, 13\}$
- $\mathcal{E}_2^{[29]} = \mathcal{E}_2^{[16]}$ ✗
- $\mathcal{E}_2^{[30]} = \mathcal{E}_2^{[18]}$ ✗

- $\mathcal{E}_2^{[31]} = \mathcal{E}_2^{[19]} + \{6, 11\}$
- $\mathcal{E}_2^{[32]} = \mathcal{E}_2^{[19]}$ ✗
- $\mathcal{E}_2^{[33]} = \mathcal{E}_2^{[20]}$ ✗
- $\mathcal{E}_2^{[34]} = \mathcal{E}_2^{[22]}$ ✗
- $\mathcal{E}_2^{[35]} = \mathcal{E}_2^{[23]}$ ✗
- $\mathcal{E}_2^{[36]} = \mathcal{E}_2^{[24]}$ ✗
- $\mathcal{E}_2^{[37]} = \mathcal{E}_2^{[28]}$ ✗
- $\mathcal{E}_2^{[38]} = \mathcal{E}_2^{[21]}$ ✗

6.7 $K_{4,5} - 4K_2$

The graph $K_{4,5} - 4K_2$ is internally 4-connected and thus it should be possible to carry out the same computations. In fact, we devoted a significant effort as well as computational power to this graph, but it turns out to be notably different from the other cases. The graph allows many internally 4-connected graphs to be created, which quickly become very large and thus it is very time-consuming to search for the forbidden minors in them using an exponential algorithm. Within our previous work, we managed to process graphs of size approximately 11 vertices. We managed to improve this bound within tool to generate results for [7], where we were able to process graphs up to size of 14–15 vertices. Using our optimized tools, we are nowadays able to process graphs with 20 vertices, but it still is not sufficient.

For the graphs generated from $K_{4,5} - 4K_2$ the sets of forbidden minors ϱ_6 corresponding with the rest of the already finished computations would be

$$\varrho_6 = \Lambda \cup \Delta\mathrm{Y}(K_7 - C_4) \cup \Delta\mathrm{Y}(K_{1,2,2,2}) \cup \{K_{4,4} - e\}$$

The following Theorem 6.1 can be considered proven at this point:

Theorem 6.1. *There is only a finite set $\mathcal{U}$, a subset of*

$$\mathcal{V} = \{G \mid G \in \mathcal{A}_H \text{ for } H \in (\Delta\mathrm{Y}(K_{1,2,2,2}) \cup \{K_{4,4} - e\})\},$$

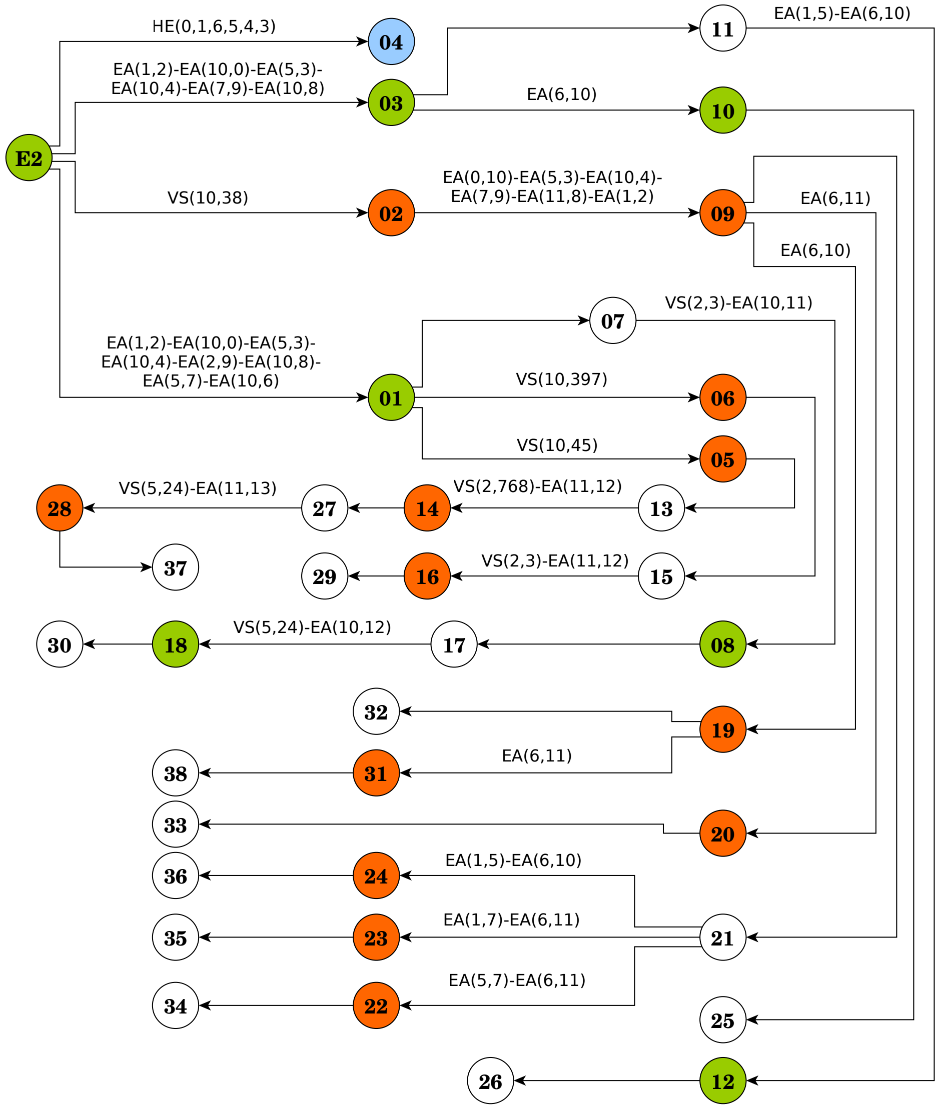

Figure 6.7: A tree depicting the generating process for the graph $\mathcal{E}_2$.

of non-projective graphs that are internally 4-connected, have a finite planar emulator and do not contain any minor isomorphic to a member of $K_7 - C_4$ family or $K_{4,5} - 4K_2$. Consequently, if a non-projective graph G has a finite planar emulator, it is a planar expansion of a member of $\mathcal{U}$, or it contains a minor isomorphic to a member of the $K_7 - C_4$ family or to $K_{4,5} - 4K_2$.

6.8 Analysis of the generated graphs

In the previous sections, we described how we generated a set of internally 4-connected graphs that are, up to several cases, the only important ones for the problem of finite planar emulators. In this section, we would like to analyse these results with respect to the structural relations among them. We also speak about the phenomenon of violating minors and an impact that it has has on the potentially planar-emulable graphs.

Recall that existence of finite planar emulators is closed under taking minors. The trees of the splitting operations that lead to generating the graphs show minor relations among these graphs, however those are surely not the only ones. As shown by the series of searches that we have conducted so far (starting with [6]), there are many ways how those graphs can be obtained one from another. The order in which we we obtained the graphs is simply given by the way how the iterators in the particular implemented tool behave. Thus we can see that there are many more minor relations among the results and they main be used to pinpoint the most important graphs that we would wish to understand better.

Let us focus on the graphs that were obtained from the family of $K_{1,2,2,2}$. We generated the ordering by $\leq$ relation, although we do not list the complete structure here as the ordering is rather dense. Instead, we include the structure in combination with analysis based on the violating minors of $\mathcal{E}_2$.

Our computation showed that any graph G that has $\mathcal{E}_2$ minor, does not have a minor from the family of $K_7 - C_4$ or $K_{4,5} - 4K_2$ and is distinct from $\mathcal{E}_2^{[0]}, \mathcal{E}_2^{[2]}, \mathcal{E}_2^{[4]}$ has a violating split graph H. Recall that this means that there is a vertex split of G that results in graph H that is not internally 4-connected. We verified that by removing all the violating edges, and consequently also edges that whose endvertices share a common cubic neighbour, we obtain graph F which isomorphic to a subdivision of some other graph $\mathcal{E}_2^{[x]}$. This is an extremely interesting phenomenon as it allows us to take an emulator of $\mathcal{E}_2^{[x]}$, subdivide the edges in order to obtain emulator of graph F, and iteratively add the edges violating edges (in the backward order as they were removed from H to obtain F) adjusting cubic degree vertices in the emulator whenever necessary. This process leads to constructing an emulator of H and by reverting the vertex split, also to emulator of G. Thus, the following is corollary of existence of violating splits.

Corollary 6.2. *Let G be an internally 4-connected graph, $H = G < v\{{N_1 \atop N_2}\}$ graph that is not internally 4-connected and F graph obtained from H by iterative removal of violating edges, and consequently all the edges that connect two neighbours of a cubic vertex, from H. Furthermore, let F be homeomorphic to a graph $F_0 \leq G$. Then, for two graphs $J_1, J_2 \in \{F_0, F, G, H\}$, J_1 has a finite planar emulator if and only if J_2 has a finite planar emulator.*

Let us remark Corollary 6.2 can be formulated in the same way for finite planar covers.

We can observe three sets of graphs that contain $\mathcal{E}_2$ minor graphs in which are equivalent with respect to the existence of finite planar emulators:

- $\mathcal{IE}_2^{[0]} = \{\mathcal{E}_2^{[0]}, \mathcal{E}_2^{[1]}, \mathcal{E}_2^{[3]}, \mathcal{E}_2^{[8]}, \mathcal{E}_2^{[10]}, \mathcal{E}_2^{[12]}, \mathcal{E}_2^{[18]}\}$
- $\mathcal{IE}_2^{[2]} = \{\mathcal{E}_2^{[2]}, \mathcal{E}_2^{[5]}, \mathcal{E}_2^{[6]}, \mathcal{E}_2^{[9]}, \mathcal{E}_2^{[14]}, \mathcal{E}_2^{[16]}, \mathcal{E}_2^{[19]}, \mathcal{E}_2^{[20]}, \mathcal{E}_2^{[22..24]}, \mathcal{E}_2^{[28]}, \mathcal{E}_2^{[31]}\}$
- $\mathcal{IE}_2^{[4]} = \{\mathcal{E}_2^{[4]}\}$

Furthermore, as there is a graph G in $\mathcal{IE}_2^{[0]}$ for which a graph H in $\mathcal{IE}_2^{[2]}$ such that G is a minor of H exists, if a graph in $\mathcal{IE}_2^{[2]}$ has finite planar emulators (thus all the graphs in $\mathcal{IE}_2^{[2]}$ do as well), all the graphs from $\mathcal{IE}_2^{[0]}$ are emulable in planar too. The same can be observed with with respect to $\mathcal{IE}_2^{[4]}$. Hence, we can in some sense say that these classes tend to form a hierarchical structure where $\mathcal{IE}_2^{[0]} \preceq \mathcal{IE}_2^{[2]}$ and $\mathcal{IE}_2^{[0]} \preceq \mathcal{IE}_2^{[4]}$. Also note that we already have finite planar emulator for $\mathcal{E}_2^{[0]} \simeq \mathcal{E}_2$ and thus for the entire class $\mathcal{IE}_2^{[0]}$.

The particular violating splits for each graph are listed below. Note that for some of the graphs, multiple violating splits exists. In such a case however, all such splits lead upon to different subdivisions of the same graph.

- $\mathcal{E}_2^{[1]} < 0\left\{\begin{smallmatrix}1,2\\3,10\end{smallmatrix}\right\}\begin{array}{l}-\{1,2\}-\{3,10\}-\{4,5\}-\{6,10\}\\-\{9,10\}-\{2,8\}-\{5,7\}-\{7,10\}\end{array}$
- $\mathcal{E}_2^{[3]} < 1\left\{\begin{smallmatrix}0,2\\6,10\end{smallmatrix}\right\}\begin{array}{l}-\{0,2\}-\{3,10\}-\{4,5\}\\-\{8,10\}-\{9,10\}-\{7,8\}\end{array}$
- $\mathcal{E}_2^{[5]} < 1\left\{\begin{smallmatrix}0,2\\6,10\end{smallmatrix}\right\}\begin{array}{l}-\{0,2\}-\{3,11\}-\{4,5\}-\{6,10\}\\-\{9,10\}-\{2,8\}-\{5,7\}-\{7,10\}\end{array}$
- $\mathcal{E}_2^{[5]} < 6\left\{\begin{smallmatrix}5,7\\1,10\end{smallmatrix}\right\}\begin{array}{l}-\{1,10\}-\{5,7\}-\{8,10\}-\{0,2\}\\-\{2,9\}-\{3,11\}-\{4,5\}-\{4,10\}\end{array}$
- $\mathcal{E}_2^{[6]} < 0\left\{\begin{smallmatrix}1,2\\3,11\end{smallmatrix}\right\}\begin{array}{l}-\{1,2\}-\{3,11\}-\{4,5\}-\{6,10\}\\-\{9,10\}-\{2,8\}-\{5,7\}-\{7,11\}\end{array}$
- $\mathcal{E}_2^{[6]} < 10\left\{\begin{smallmatrix}1,6\\4,5,9,11\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{1,6\}-\{5,7\}-\{8,11\}-\{0,2\}\\-\{2,9\}-\{3,11\}-\{4,5\}-\{4,10\}\end{array}$
- $\mathcal{E}_2^{[8]} < 0\left\{\begin{smallmatrix}3,10\\1,11\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{1,11\}-\{2,10\}-\{3,10\}-\{4,5\}-\{6,10\}\\-\{8,9\}-\{9,10\}-\{5,7\}-\{7,10\}\end{array}$
- $\mathcal{E}_2^{[8]} < 2\left\{\begin{smallmatrix}8,9\\10,11\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{8,9\}-\{10,11\}-\{0,1\}-\{3,10\}-\{4,5\}\\-\{4,10\}-\{6,10\}-\{7,10\}-\{5,6\}\end{array}$
- $\mathcal{E}_2^{[8]} < 5\left\{\begin{smallmatrix}3,4\\6,7,10\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{3,4\}-\{9,10\}-\{0,10\}-\{1,11\}-\{2,8\}\\-\{2,10\}-\{6,10\}-\{7,10\}-\{5,6\}\end{array}$
- $\mathcal{E}_2^{[9]} < 5\left\{\begin{smallmatrix}3,4\\6,11\end{smallmatrix}\right\}\begin{array}{l}-\{3,4\}-\{9,10\}-\{0,10\}\\-\{1,2\}-\{7,8\}-\{8,11\}\end{array}$
- $\mathcal{E}_2^{[9]} < 7\left\{\begin{smallmatrix}8,9\\6,10\end{smallmatrix}\right\}\begin{array}{l}-\{8,9\}-\{2,11\}-\{4,10\}\\-\{0,1\}-\{3,5\}-\{3,10\}\end{array}$
- $\mathcal{E}_2^{[9]} < 11\left\{\begin{smallmatrix}2,8\\1,5,10\end{smallmatrix}\right\}\begin{array}{l}-\{2,8\}-\{7,9\}-\{0,1\}\\-\{3,10\}-\{4,5\}-\{4,10\}\end{array}$
- $\mathcal{E}_2^{[10]} < 0\left\{\begin{smallmatrix}1,2\\3,10\end{smallmatrix}\right\}\begin{array}{l}-\{1,2\}-\{3,10\}-\{4,5\}-\{6,10\}\\-\{8,10\}-\{9,10\}-\{7,8\}\end{array}$
- $\mathcal{E}_2^{[12]} < 7\left\{\begin{smallmatrix}8,9\\6,10\end{smallmatrix}\right\}\begin{array}{l}-\{6,10\}-\{8,9\}-\{1,5\}-\{2,10\}\\-\{4,10\}-\{0,1\}-\{3,5\}-\{3,10\}\end{array}$
- $\mathcal{E}_2^{[14]} < 1\left\{\begin{smallmatrix}0,2\\6,10\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{0,2\}-\{3,11\}-\{4,5\}-\{6,10\}-\{9,10\}\\-\{11,12\}-\{5,7\}-\{8,9\}-\{8,10\}\end{array}$
- $\mathcal{E}_2^{[14]} < 9\left\{\begin{smallmatrix}4,10\\8,12\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{4,10\}-\{8,12\}-\{2,11\}-\{3,5\}-\{7,10\}\\-\{0,1\}-\{0,11\}-\{5,6\}-\{6,10\}\end{array}$
- $\mathcal{E}_2^{[14]} < 8\left\{\begin{smallmatrix}7,10\\9,12\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{7,10\}-\{9,12\}-\{2,11\}-\{4,10\}-\{5,6\}\\-\{0,1\}-\{1,10\}-\{3,5\}-\{3,11\}\end{array}$
- $\mathcal{E}_2^{[14]} < 12\left\{\begin{smallmatrix}8,9\\2,11\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{2,11\}-\{8,9\}-\{0,1\}-\{3,11\}-\{4,5\}\\-\{4,10\}-\{6,10\}-\{7,10\}-\{5,6\}\end{array}$
- $\mathcal{E}_2^{[14]} < 6\left\{\begin{smallmatrix}5,7\\1,10\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{1,10\}-\{5,7\}-\{8,10\}-\{9,12\}-\{0,2\}\\-\{2,11\}-\{3,11\}-\{4,5\}-\{4,10\}\end{array}$
- $\mathcal{E}_2^{[16]} < 0\left\{\begin{smallmatrix}3,11\\1,12\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{1,12\}-\{2,11\}-\{3,11\}-\{4,5\}-\{6,10\}\\-\{8,9\}-\{9,10\}-\{5,7\}-\{7,11\}\end{array}$
- $\mathcal{E}_2^{[16]} < 2\left\{\begin{smallmatrix}8,9\\11,12\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{8,9\}-\{11,12\}-\{0,1\}-\{3,11\}-\{4,5\}\\-\{4,10\}-\{6,10\}-\{7,11\}-\{5,6\}\end{array}$
- $\mathcal{E}_2^{[16]} < 4\left\{\begin{smallmatrix}3,5\\9,10\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{3,5\}-\{9,10\}-\{0,11\}-\{1,12\}-\{2,8\}\\-\{2,11\}-\{6,10\}-\{7,11\}-\{5,6\}\end{array}$
- $\mathcal{E}_2^{[16]} < 10\left\{\begin{smallmatrix}1,6\\4,5,9,11\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{1,6\}-\{5,7\}-\{8,11\}-\{0,12\}-\{2,9\}\\-\{2,11\}-\{3,11\}-\{4,5\}-\{4,10\}\end{array}$
- $\mathcal{E}_2^{[18]} < 0\left\{\begin{smallmatrix}3,10\\1,11\end{smallmatrix}\right\}$
  $\begin{array}{l}-\{1,11\}-\{2,10\}-\{3,10\}-\{4,12\}-\{5,10\}\\-\{6,7\}-\{6,10\}-\{8,9\}-\{8,10\}-\{9,10\}\end{array}$
- $\mathcal{E}_2^{[19]} < 7\left\{\begin{smallmatrix}8,9\\6,10\end{smallmatrix}\right\}\begin{array}{l}-\{6,10\}-\{8,9\}-\{2,11\}-\{4,10\}\\-\{0,1\}-\{3,5\}-\{3,10\}\end{array}$
- $\mathcal{E}_2^{[19]} < 3\left\{\begin{smallmatrix}4,5\\0,10\end{smallmatrix}\right\}\begin{array}{l}-\{0,10\}-\{1,2\}-\{4,5\}-\{8,11\}\\-\{9,10\}-\{7,8\}-\{6,10\}\end{array}$
- $\mathcal{E}_2^{[19]} < 8\left\{\begin{smallmatrix}7,9\\2,11\end{smallmatrix}\right\}\begin{array}{l}-\{2,11\}-\{7,9\}-\{0,1\}-\{3,10\}\\-\{4,5\}-\{4,10\}-\{6,10\}\end{array}$
- $\mathcal{E}_2^{[20]} < 6\left\{\begin{smallmatrix}1,7\\5,11\end{smallmatrix}\right\}\begin{array}{l}-\{5,11\}-\{3,4\}-\{9,10\}-\{0,10\}\\-\{1,2\}-\{7,8\}-\{8,11\}\end{array}$
- $\mathcal{E}_2^{[20]} < 8\left\{\begin{smallmatrix}7,9\\2,11\end{smallmatrix}\right\}\begin{array}{l}-\{2,11\}-\{7,9\}-\{0,1\}-\{3,10\}\\-\{4,5\}-\{4,10\}-\{6,11\}\end{array}$
- $\mathcal{E}_2^{[20]} < 7\left\{\begin{smallmatrix}8,9\\6,10\end{smallmatrix}\right\}\begin{array}{l}-\{8,9\}-\{2,11\}-\{4,10\}-\{0,1\}\\-\{3,5\}-\{3,10\}-\{6,11\}\end{array}$
- $\mathcal{E}_2^{[22]} < 2\left\{\begin{smallmatrix}0,1\\8,11\end{smallmatrix}\right\}\begin{array}{l}-\{0,1\}-\{3,10\}-\{4,5\}-\{6,11\}\\-\{8,11\}-\{9,10\}-\{5,7\}-\{7,8\}\end{array}$
- $\mathcal{E}_2^{[22]} < 8\left\{\begin{smallmatrix}7,9\\2,11\end{smallmatrix}\right\}\begin{array}{l}-\{2,11\}-\{7,9\}-\{0,1\}-\{3,10\}\\-\{4,5\}-\{4,10\}-\{6,11\}-\{5,7\}\end{array}$
- $\mathcal{E}_2^{[22]} < 7\left\{\begin{smallmatrix}8,9\\5,6,10\end{smallmatrix}\right\}\begin{array}{l}-\{8,9\}-\{2,11\}-\{4,10\}-\{0,1\}\\-\{3,5\}-\{3,10\}-\{6,11\}-\{5,7\}\end{array}$
- $\mathcal{E}_2^{[23]} < 5\left\{\begin{smallmatrix}3,4\\6,11\end{smallmatrix}\right\}\begin{array}{l}-\{3,4\}-\{6,11\}-\{9,10\}-\{0,10\}\\-\{1,2\}-\{1,7\}-\{7,8\}-\{8,11\}\end{array}$
- $\mathcal{E}_2^{[23]} < 8\left\{\begin{smallmatrix}7,9\\2,11\end{smallmatrix}\right\}\begin{array}{l}-\{2,11\}-\{7,9\}-\{0,1\}-\{3,10\}\\-\{4,5\}-\{4,10\}-\{6,11\}-\{1,7\}\end{array}$
- $\mathcal{E}_2^{[23]} < 7\left\{\begin{smallmatrix}8,9\\1,6,10\end{smallmatrix}\right\}\begin{array}{l}-\{8,9\}-\{2,11\}-\{4,10\}-\{0,1\}\\-\{3,5\}-\{3,10\}-\{6,11\}-\{1,7\}\end{array}$

- $\mathcal{E}_2^{[24]} < 0\left\{\begin{smallmatrix}1,2\\3,10\end{smallmatrix}\right\}\begin{smallmatrix}-\{1,2\}-\{3,10\}-\{4,5\}-\{8,11\}\\-\{9,10\}-\{7,8\}-\{6,10\}-\{1,5\}\end{smallmatrix}$
- $\mathcal{E}_2^{[24]} < 7\left\{\begin{smallmatrix}8,9\\6,10\end{smallmatrix}\right\}\begin{smallmatrix}-\{6,10\}-\{8,9\}-\{1,5\}-\{2,11\}\\-\{4,10\}-\{0,1\}-\{3,5\}-\{3,10\}\end{smallmatrix}$
- $\mathcal{E}_2^{[24]} < 8\left\{\begin{smallmatrix}7,9\\2,11\end{smallmatrix}\right\}\begin{smallmatrix}-\{2,11\}-\{7,9\}-\{0,1\}-\{3,10\}\\-\{4,5\}-\{4,10\}-\{6,10\}-\{1,5\}\end{smallmatrix}$
- $\mathcal{E}_2^{[28]} < 2\left\{\begin{smallmatrix}0,1\\11,12\end{smallmatrix}\right\}$
  $\begin{smallmatrix}-\{0,1\}-\{3,11\}-\{4,13\}-\{5,11\}-\{6,7\}\\-\{6,10\}-\{8,10\}-\{9,10\}-\{9,12\}-\{11,12\}\end{smallmatrix}$
- $\mathcal{E}_2^{[28]} < 10\left\{\begin{smallmatrix}1,6\\4,7,8,9,11\end{smallmatrix}\right\}$
  $\begin{smallmatrix}-\{1,6\}-\{5,7\}-\{8,10\}-\{9,12\}-\{11,13\}\\-\{0,2\}-\{2,11\}-\{3,4\}-\{3,11\}-\{4,10\}\end{smallmatrix}$
- $\mathcal{E}_2^{[28]} < 10\left\{\begin{smallmatrix}7,8\\1,4,6,9,11\end{smallmatrix}\right\}$
  $\begin{smallmatrix}-\{7,8\}-\{9,12\}-\{2,11\}-\{4,10\}-\{5,6\}\\-\{11,13\}-\{0,1\}-\{1,10\}-\{3,4\}-\{3,11\}\end{smallmatrix}$
- $\mathcal{E}_2^{[31]} < 0\left\{\begin{smallmatrix}1,2\\3,10\end{smallmatrix}\right\}\begin{smallmatrix}-\{1,2\}-\{3,10\}-\{4,5\}-\{6,11\}\\-\{8,11\}-\{9,10\}-\{7,8\}-\{6,10\}\end{smallmatrix}$
- $\mathcal{E}_2^{[31]} < 7\left\{\begin{smallmatrix}8,9\\6,10\end{smallmatrix}\right\}\begin{smallmatrix}-\{6,10\}-\{8,9\}-\{2,11\}-\{4,10\}\\-\{0,1\}-\{3,5\}-\{3,10\}-\{6,11\}\end{smallmatrix}$
- $\mathcal{E}_2^{[31]} < 8\left\{\begin{smallmatrix}7,9\\2,11\end{smallmatrix}\right\}\begin{smallmatrix}-\{2,11\}-\{7,9\}-\{0,1\}-\{3,10\}\\-\{4,5\}-\{4,10\}-\{6,10\}-\{6,11\}\end{smallmatrix}$

Since the violating splits effectively decompose the set $\mathcal{A}_{\mathcal{E}_2}$ into three classes, we can raise question what happens with the graphs generated from graphs in $\Delta\mathrm{Y}(K_{1,2,2,2})$. The only other violating splits that occur within graphs from $\Delta\mathrm{Y}(K_{1,2,2,2})$ are in the case of $\mathcal{D}_2$. These splits were also discovered within [20].

- $\mathcal{D}_2^{[3]} < 0\left\{\begin{smallmatrix}1,3\\6,9\end{smallmatrix}\right\}\begin{smallmatrix}-\{1,3\}-\{2,9\}-\{4,7\}\\-\{6,9\}-\{8,9\}-\{5,6\}\end{smallmatrix}$ leading to a subdivision of $\mathcal{D}_2^{[0]} \simeq \mathcal{D}_2$
- $\mathcal{D}_2^{[8]}$ with the vertex splits (listed below) leading to a subdivision of $\mathcal{D}_2^{[1]}$:
  - $\mathcal{D}_2^{[8]} < 7\left\{\begin{smallmatrix}2,4\\8,9\end{smallmatrix}\right\}\begin{smallmatrix}-\{2,4\}-\{8,9\}-\{1,10\}\\-\{5,6\}-\{0,3\}-\{0,9\}\end{smallmatrix}$
  - $\mathcal{D}_2^{[8]} < 0\left\{\begin{smallmatrix}1,3\\6,9\end{smallmatrix}\right\}\begin{smallmatrix}-\{1,3\}-\{2,10\}-\{4,7\}\\-\{6,9\}-\{8,9\}-\{5,6\}\end{smallmatrix}$
  - $\mathcal{D}_2^{[8]} < 8\left\{\begin{smallmatrix}5,6\\7,9\end{smallmatrix}\right\}\begin{smallmatrix}-\{5,6\}-\{7,9\}-\{0,9\}\\-\{1,3\}-\{2,4\}-\{2,10\}\end{smallmatrix}$
  - $\mathcal{D}_2^{[8]} < 6\left\{\begin{smallmatrix}5,8\\0,9\end{smallmatrix}\right\}\begin{smallmatrix}-\{0,9\}-\{1,3\}-\{2,10\}\\-\{4,7\}-\{5,8\}-\{7,9\}\end{smallmatrix}$
  - $\mathcal{D}_2^{[8]} < 9\left\{\begin{smallmatrix}7,8\\0,4,6,10\end{smallmatrix}\right\}\begin{smallmatrix}-\{7,8\}-\{2,4\}-\{5,6\}\\-\{0,9\}-\{1,3\}-\{1,10\}\end{smallmatrix}$

Generating internally 4-connected graph that contain $\mathcal{D}_2$ minor, both these cycles can be avoided as the we can apply Theorem 3.20. Also, note that the graphs $\mathcal{D}_2^{[3]}$ and $\mathcal{D}_2^{[0]}$ are minors of some graphs in the set $\mathcal{IE}_2^{[0]}$, and $\mathcal{D}_2^{[8]}$ together with $\mathcal{D}_2^{[1]}$ are minors of some graphs in the set $\mathcal{IE}_2^{[2]}$. Violating minors occur also in the case of $K_{4,4}-e$ — they are listed in Appendix A.4.

As planar-emulable graphs are closed under taking minors, we also exhaustively enumerated the ordering of all the graphs produced from $\Delta\mathrm{Y}(K_{1,2,2,2})$ under the minor relation. This ordering decomposes the set into only seven classes $\mathcal{JG}$ for which the following holds: if $\mathcal{G}$ is a planar-emulable graph, then all the graphs in $\mathcal{JG}$ are are planar-emulable as well. Due to the way how violating splits behave on the graphs generated from $\mathcal{E}_2$, we can include all the minors of $\mathcal{JG}$ for $\mathcal{G}$ being a graph from $\mathcal{IE}_2^{[0]}, \mathcal{IE}_2^{[2]}, \mathcal{IE}_2^{[4]}$ into one class obtaining the following list.

- $\mathcal{JE}_2^{[0]} = \mathcal{IE}_2^{[0]} \cup \{ K_{1,2,2,2}^{[0]}, \mathcal{B}_7^{[0..1]}, \mathcal{B}_7^{[7]}, \mathcal{C}_3^{[0]}, \mathcal{C}_3^{[3]}, \mathcal{C}_3^{[11]}, \mathcal{C}_3^{[19..20]}, \mathcal{D}_2^{[0]}, \mathcal{D}_2^{[2..3]}, \mathcal{D}_2^{[11]}, \mathcal{D}_2^{[18]}, \mathcal{D}_2^{[27]}, \mathcal{D}_2^{[28]} \}$

- $\mathcal{JE}_2^{[2]} = \mathcal{JE}_2^{[0]} \cup \mathcal{IE}_2^{[2]} \cup \{ K_{1,2,2,2}^{[2]}, \mathcal{B}_7^{[4]}, \mathcal{B}_7^{[6]}, \mathcal{B}_7^{[8]}, \mathcal{B}_7^{[10..12]}, \mathcal{C}_3^{[1]}, \mathcal{C}_3^{[2]}, \mathcal{C}_3^{[6..10]} \mathcal{C}_3^{[12..18]}, \mathcal{C}_3^{[21..30]}, \mathcal{D}_2^{[0]}, \mathcal{D}_2^{[1]}, \mathcal{D}_2^{[7..10]}, \mathcal{D}_2^{[13..17]}, \mathcal{D}_2^{[20..26]}, \mathcal{D}_2^{[30..39]} \}$

- $\mathcal{JE}_2^{[4]} = \mathcal{IE}_2^{[4]} \cup \{\mathcal{C}_4^{[0]}, \mathcal{C}_4^{[2]}, \mathcal{C}_3^{[0]}, \mathcal{D}_2^{[0]}, \mathcal{D}_2^{[5]}, \mathcal{E}_2^{[0]}\}$

- $\mathcal{JD}_2^{[29]} = \{\mathcal{B}_7^{[0..2]}, \mathcal{C}_3^{[0]}, \mathcal{C}_3^{[3]}, \mathcal{C}_3^{[4]}, \mathcal{C}_3^{[11]}, \mathcal{C}_4^{[0..3]}, \mathcal{C}_4^{[5]}, \mathcal{C}_4^{[7]}, \mathcal{D}_2^{[0]}, \mathcal{D}_2^{[2]}, \mathcal{D}_2^{[5]}, \mathcal{D}_2^{[6]}, \mathcal{D}_2^{[11..12]}, \mathcal{D}_2^{[18..19]}, \mathcal{D}_2^{[29]}\}$

- $\mathcal{JC}_4^{[6]} = \{\mathcal{B}_7^{[0]}, \mathcal{B}_7^{[2]}, \mathcal{C}_4^{[0]}, \mathcal{C}_4^{[2]}, \mathcal{C}_4^{[6]}, \mathcal{C}_3^{[0]}, \mathcal{C}_3^{[4]}\}$

- $\mathcal{JC}_4^{[4]} = \{\mathcal{C}_4^{[0]}, \mathcal{C}_4^{[1]}, \mathcal{C}_4^{[4]}\}$

- $\mathcal{JB}_7^{[9]} = \{\mathcal{B}_7^{[0]}, \mathcal{B}_7^{[2]}, \mathcal{B}_7^{[4]}, \mathcal{B}_7^{[9]}\}$

From this perspective, the most interesting graphs for planar emulators are the graphs $\mathcal{E}_2^{[4]}, \mathcal{D}_2^{[29]}, \mathcal{C}_4^{[6]}, \mathcal{C}_4^{[4]}, \mathcal{B}_7^{[9]}$ as these graphs contain all the other graphs from the class as minors, and any graph from $\mathcal{E}_2^2$. Note that the classes are not pairwise disjoint. Emulators for the entire class $\mathcal{E}_2^{[0]}$ can be derived from our emulator for $\mathcal{E}_2$, and finite planar emulator for $\mathcal{C}_4$ is known as well.

None of the internally 4-connected graphs with a minor isomorphic to a graph in $\Delta\mathrm{Y}(K_{1,2,2,2})$ rejected due to some forbidden minor M was rejected because of M isomorphic to $K_{4,5} - 4K_2$. Thus for all ϱ_i with $1 \leq i \leq 6$, it is not necessary that $K_{4,5} - 4K_2$ is included as a forbidden minor. Much stronger consequence can be however observed from the other point of view:

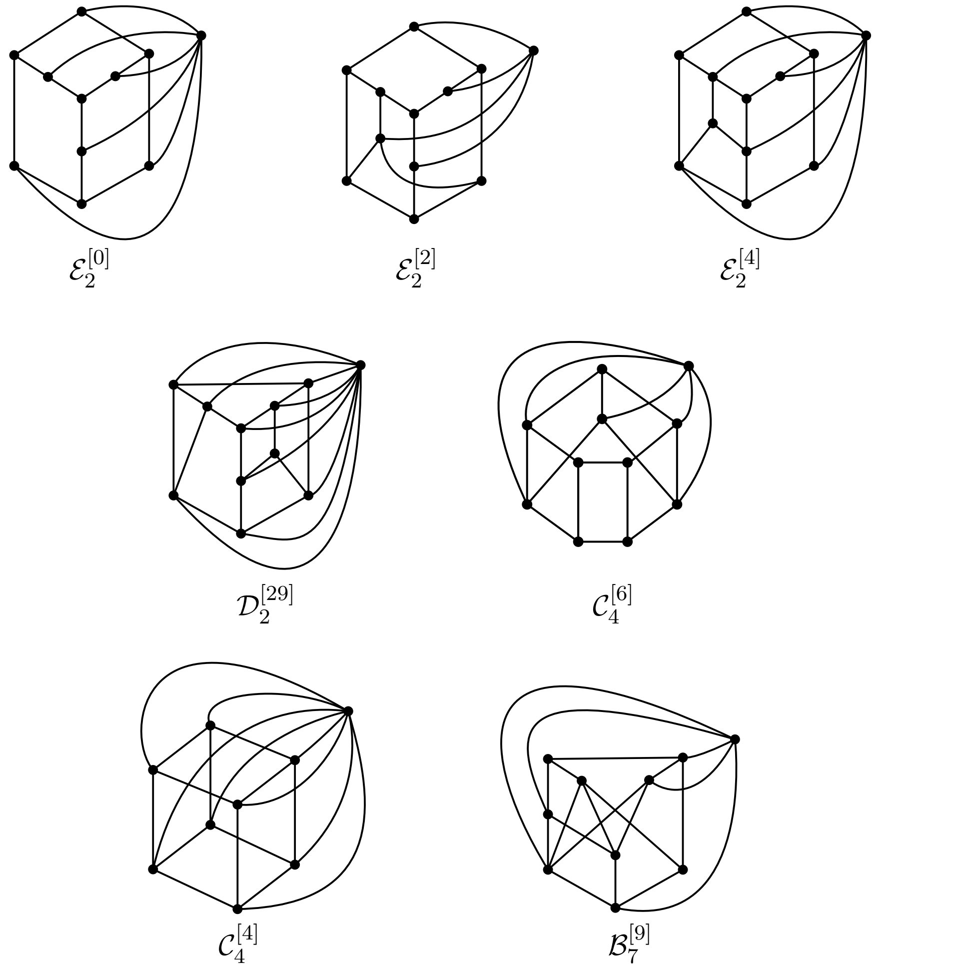

Figure 6.8: The graphs that together contain minors isomorphic to all internally 4-connected graphs generated from the family of $K_{1,2,2,2}$.

Lemma 6.3. *Let G be an internally 4-connected graph that contains a minor isomorphic to $K_{4,5} - 4K_2$ and has no minor from $\Delta Y(K_7 - C_4)$ or $K_{4,4} - e$ and does not contain any minor forbidden for the finite planar emulators. Then, G does not contain a minor from the family of $K_{1,2,2,2}$.*

Let us remark that no similar conclusion can be made with respect to the graph $K_{4,4} - e$ as our computations discovered internally 4-connected graphs that contain both $K_{4,4} - e$ and $K_{4,5} - 4K_2$ minors and are potentially emulable in planar.

Some of those graphs were listed in [7], where we called the listing complete. We unfortunately have to make a correction to that statement due to a mistake in the structure of the graph $\mathcal{E}_{11}$ as published in [18]. The list provided in [7] is incomplete and there are more internally 4-connected graphs that can be obtained from $K_{4,4} - e$ and contain minor isomorphic to $K_{4,5} - 4K_2$ and no minor from $\varrho_0 \setminus \{K_{4,5} - 4K_2\}$. The remaining results related to $K_{4,4} - e$ listed in this work, in [6] and [7] remain valid. The wrong structure did not affect previously published results for any other graph.

Chapter 7

Cubic graphs

While characterization of planar-emulable graphs has proven itself to be difficult in general, significant progress can be made in a special case. Negami's conjecture has been confirmed in the case of cubic graphs in [28], and the same readily follows from [20]. Here we prove:

Theorem 7.1. *If a cubic non-projective graph H has a finite planar emulator, then H is a planar expansion (Definition 3.15) of one of two minimal cubic non-projective graphs shown in Figure 7.1.*

The purpose of this section is to prove Theorem 7.1. As approach to Theorem 7.1 slightly differs from the previous rather algorithmic chapters of this work, we restate the folklore known facts about planar-emulable graphs that are important in this chapter within Proposition 7.2.

Proposition 7.2. *Let G be a connected graph.*

1. *The class of planar-emulable graphs is closed under taking minors.*
2. *If G is projective, then G has a finite planar emulator in form of its finite planar cover.*
3. *If G contains two disjoint k-graphs or a $K_{3,5}$ minor, then G is not planar-emulable.*
4. *G is planar-emulable if, and only if, so is any planar expansion of G.*

A computerized search for all possible counterexamples to Conjecture 3.3, carried out so far (see previous chapters), shows that a non-projective planar-emulable graph G cannot be cubic, unless G contains a minor isomorphic to $K_{4,5} - 4K_2$, or a member of the $K_7 - C_4$ family (see Chapter 6). Our new approach, Theorem 7.1, actually dismisses the former two possibilities completely and strongly restricts the latter one.

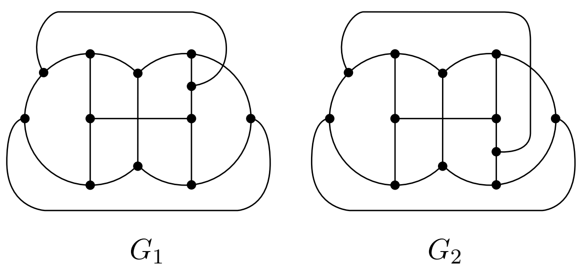

Figure 7.1: Two (out of six in total) cubic irreducible obstructions for the projective plane [16]. Although these graphs result by splitting non-projective graphs for which we have finite planar emulators [4] (namely $K_7 - C_4$ and its "relatives"), it is still open whether they are planar-emulable.

Proof of Theorem 7.1. Glover and Huneke [16] characterized the cubic graphs with projective embedding using six minimal forbidden cubic topological minors (see Figure 7.1 for two of them).

Theorem 7.3 (Glover–Huneke [16]). *There is a set $\mathcal{I}$ of six cubic graphs such that; if H is a cubic graph that does not embed in the projective plane, then H contains a graph $G \in \mathcal{I}$ as a topological minor.*

Let us point out that four out of the six graphs in $\mathcal{I}$ contain two disjoint k-graphs, and so only the remaining two— $G_1 \in \mathcal{I}$ and $G_2 \in \mathcal{I}$ of Figure 7.1, can potentially be planar-emulable. Hence the cubic graph H in Theorem 7.1 contains one of G_1, G_2 as a topological minor. In other words, there is a subgraph $G' \subseteq H$ being a subdivision of a cubic $G \in \{G_1, G_2\}$.

At this point, recall the notion of bridges introduced in Chapter 5, Section 5.5.1. A *bridge of* G' in H any connected component B of $H - V(G')$ together will all the incident edges. In a degenerate case, B might consist just of one edge from $E(H) \setminus E(G')$ with both ends in G'. We would like, for simplicity, to speak about positions of bridges with respect to the underlying cubic graph G: Such a bridge B connects to vertices u of G' which subdivide edges f of G—this is due to the cubic degree bound, and we (with negligible abuse of terminology) say that B *attaches to* this edge f in G itself.

A bridge B is *nontrivial* if B attaches to some two nonadjacent edges of

G, and B is *trivial* otherwise. For a trivial bridge B; either B attaches to only one edge in G, and we say *exclusively*, or all the edges to which B attaches in G have a vertex w in common (since G contains no triangle), and we say that B *attaches to* this w.

We divide the rest of the proof into two main cases; that either some bridge of G' in H is nontrivial or all such bridges are trivial. We moreover assume that $G' \subseteq H$ being a subdivision of G is chosen such that it has a nontrivial bridge if possible. In the "all-trivial" case one more technical condition has to be observed: Suppose B_1, B_2 are bridges such that B_1 attaches to w and B_2 attaches to an edge f incident to w in G (perhaps B_2 exclusively to f). On the path P_f which replaces (subdivides) f in G', suppose that B_2 connects to some vertex which is closer to w on P_f than some other vertex to which B_1 connects to. Then we *declare that B_2 attaches to w*, too. The transitive closure of declared attachment is well defined because of the following:

Lemma 7.4. *Let $G' \subseteq H$ be a subdivision of G where G, H are cubic graphs. Suppose that all bridges of G' in H are trivial, and that a bridge B_0 attaches (or, is declared to) both to w_1 and w_2, where $w_1w_2 \in E(G)$. Then there is $G'' \subseteq H$ which is a subdivision of G, too, and a nontrivial bridge of G'' in H exists.*

Proof. Let P_f be the path representing $f = w_1w_2$ in H. In the described situation, we call B_0 a *conflicting* bridge, and assume that $H - B_0$ has no conflicting bridge of G'. By the definition of declared attachment there exist vertices $u_1, u_2 \in V(P_f)$ such that the following holds for $i = 1, 2$: Either $u_i = w_i$ and B_0 attaches to at least two edges incident to w_i, or there is a bridge B_i connecting to u_i such that B_i attaches (or, is declared to) to w_i in G and B_0 connects the two components of $P_f - u_i$ together. Notice that $B_1 \neq B_2$ and u_1 is closer to w_1 on P_f than u_2 (since $H - B_0$ has no conflicting bridge).

One can now easily check that there exist two internally disjoint paths from u_i to the two neighbours of w_i not on P_f, for each $i = 1, 2$ (Figure 7.2). Hence there exists new $G'' \subseteq H$ a subdivision of G such that the vertices w_1, w_2 now correspond to u_1, u_2, respectively, and the bridge of G'' arising from B_0 is nontrivial. $\square$

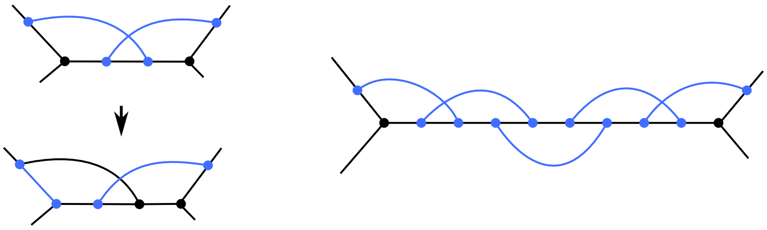

Figure 7.2: Illustration for sketch proof of Lemma 7.4. The trivial bridge on the left takes over the role of a branch vertex of G in the graph G', resulting in existence of a nontrivial bridge. The other case shows when the transitive closure of declared attachment becomes important.

Lemma 7.5. *Let $G' \subseteq H$ be a subdivision of G where G, H are cubic non-projective graphs and G does not contain two disjoint k-graphs. Suppose that all bridges of G' in H are trivial, and no one is conflicting (cf. Lemma 7.4). Then H does not contain two disjoint k-graphs if, and only if, H is a planar expansion of G.*

Proof. If H is a planar expansion of G, then two disjoint k-graphs in H would imply containment of those in G itself, which is not possible. In the converse direction, we assume that H is not a planar expansion of G. Let B_v be the union of all trivial bridges of G' in H that attach or are declared to attach to a vertex $v \in V(G)$. Let B_f be the union of all trivial bridges of G' in H that attach exclusively to an edge $f \in E(G)$. Since H is not a planar expansion of G, for at least one $x \in V(G) \cup E(G)$ the subgraph $H_x = G' \cup B_x$ is not a planar expansion of G, too. For simplicity, we consider only the more interesting case $x = u \in V(G)$. See an illustration in Figure 7.3.

Let $G'_u \subseteq G'$ denote the corresponding subdivision of $G - u$. Let $C = \{e_1, e_2, e_3\}$ be a minimal edge-cut in H_u which separates G'_u on one side and $B'_u \supset B_u \cup \{u\}$ on the other side. Then our graph H_u is not a planar expansion of G' if and only if B'_u is not planar with all the three connections to C on the outer face. The latter can be characterized by containment of a $K_{2,3}$ subdivision in B'_u with the size-three part incident to C. Then it is easy to show that $G' \cup B_u$ confirms to Definition 3.9 of two disjoint k-graphs, since $G - u$ is connected and particularly G is non-planar. $\square$

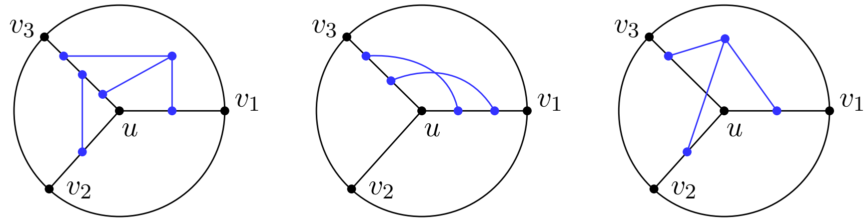

Figure 7.3: Illustration of three collections of trivial bridges that attach to a cubic vertex u. The first collection gives a planar expansion, while the other two are "minimal" non-planar-expansion cases.

Lemma 7.6. *Let $G' \subseteq H$ be a subdivision of G where G, H are cubic and G is not projective. If there exists a nontrivial bridge of G' in H, then H does not have a finite planar emulator.*

Proof. Starting from Theorem 7.3, we have exhaustively verified that for $G \in \{G_1, G_2\}$, all the graphs $G' + e$ where e is a nontrivial bridge of G do not admit existence of finite planar emulator. Up to one case, all such graphs contain two disjoint k-graphs. In the one special case, the graph $G_2' + e$ does not contain two disjoint k-graphs, but it contains a $K_{3,5}$ minor. We would like to point out that due to the necessity of $K_{3,5}$ in that one case, there is likely no simple argument summarizing the cases similarly as done in Lemma 7.5. $\square$

Theorem 7.1 is then an immediate corollary of Lemmas 7.6 and 7.5.

Chapter 8

Conclusions

By application of Theorems 3.19 and Theorem 3.20, we confirmed the previously obtained results. We also confirmed our previous suggestion that these two theorems can prove more suitable for generating internally 4-connected graphs that can have a finite planar emulator. Using optimized tools, we managed to finish the exhaustive search for the graph $\mathcal{E}_2$, which was one of the main goals of this work. Furthermore, we provided a detailed analysis of the obtained results and described an especially interesting behaviour of the vertex split operation, called violating splits, on some internally 4-connected non-projective graphs. We also considered the problem of planar-emulations for a restricted class of graphs and showed that in such a case, it becomes significantly easier leaving only two graphs in question. In this chapter, we provide an overview of our results and some suggestions for the future work.

Except for the obvious impact on planar-emulations, there is also another interesting perception of our exhaustive searches. We have shown that up to a finite number of exceptions and planar expansions of internally 4-connected graphs, the family of $K_{1,2,2,2}$ and the graph $K_{4,4}$–e can be disregarded as obstructions for the projective plane. With respect to planar emulators, we are in fact interested if there is an infinite sequence of internally 4-connected graphs without projective embedding that does not contain 2 disjoint k-graphs or a graph isomorphic $K_{3,5}$ as a minor, still leaving a question mark hanging above the family of $K_7 - C_4$. However, is it possible that the main obstructions (with finite number of exceptions) for the internally 4-connected graphs with projective embedding are only 2 disjoint k-graphs and the graph $K_{3,5}$? Or perhaps that it is only 2 disjoint k-graphs?

G. Ding provided a list of minor minimal obstructions of the internally 4-connected graphs that embed in the projective plane [9]. The list includes the graphs in $\Delta\mathrm{Y}(K_{1,2,2,2})$, $K_{4,5} - 4K_2$, $K_{4,4} - e$ and graphs that contain a minor isomorphic to $K_{3,5}$, a graph from $\Delta\mathrm{Y}(K_7 - C_4)$ and some graphs that immediately contain 2 disjoint k-graphs. Using our software tools and the notion of splitting described in this work and related literature [6, 21, 18], it might be possible to conduct a series of computations leading to a positive answer to the questions raised above. We took some preliminary steps in this direction, although the amount of effort made, regardless of the promising results, does not allow us to state a conjecture. We however consider this question extremely interesting and surely suggest this topic for future work.

The family of $K_7 - C_4$ was still not reflected in our computations, because its members are not internally 4-connected and therefore, the presented tools do not apply to them. However, the significance of these graphs for the characterization of planar-emulable non-projective graphs is obvious.

So, which tools would be suitable for tackling the cases from the family of $K_7 - C_4$? We give a very brief theoretical outline here.

Assume H is a non-projective planar emulable graph having a minor F in $K_7 - C_4$ family, and that H is minimal under planar expansions. Then two cases may occur:

- H is internally 4-connected. Then there is a practically small finite set of minor-minimal graphs H_0 that "bridge" every 3-separation of F, see [15]. These graphs H_0 can be exhaustively listed, and further generating from H_0, as in Section 3.1, can be run on them.

- H still has a non-flat 3-separation. One can then consider independently in parallel each of the separation sides (replacing the other side with a cubic vertex), again as in the internally 4-connected case.

Note that the graphs in the former case were already listed by G. Ding in [9].

Our current tools were still unable to finish the search for the graph $K_{4,5} - 4K_2$. In order to finish the computation, we can either continue optimizing the current tools for graph splitting, especially in terms of graph representation, parallelization and fast minor testing. A performance improvement can also be achieved by using a different splitter theorem for non-projective internally 4-connected graphs. In this respect, we are curiously awaiting results announced by G. Ding [10, 11].

In this work, we pointed out an interesting phenomenon of violating splits and violating minors. This can be extremely interesting especially considering Conjecture 3.2 about planar coverings. Can violating splits and minors be exploited in order to provide a proof that there is no finite planar cover for some graphs in the family of $K_{1,2,2,2}$? Specifically, is there a graph G isomorphic to a subdivision of a graph in the family of $K_{1,2,2,2}$ from which we can via a sequence of edge additions into neighbourhoods of cubic vertices, and perhaps also ΔY transformations, obtain graph H that is known to not have finite planar cover?

While our main effort (started in [7, 4]) is to provide a new finite characterization of non-projective graphs with finite planar emulators, Chapter 7 shows that the problem becomes significantly easier when only a restricted class of graphs is considered. We identified two graphs (Figure 7.1), for which the existence of a finite planar emulator now becomes extremely interesting. We would like to point out that the similarity of these two graphs suggest that if one has a finite planar emulator, the other one does as well. If we however elaborate on this idea and attempt to "unify" the graphs as depicted in Figure 8.1, we have to use a nontrivial bridge, which was shown not to be possible. Perhaps, this provides a clue that these two graphs should not be planar-emulable. Surely, providing an answer for either of these two graphs would bring a better insight to the problem of planar emulations not only for the cubic case, but also in general.

Lastly, we would like to include in our opinion a remark and suggestion by M. Fellows [14]: It is actually an interesting fact that $\mathcal{F}$-coverable and $\mathcal{F}$-emulable are general "operators" on minor ideals. If $\mathcal{F}$ is any minor ideal (e.g. in this work, $\mathcal{F}$ = planar graphs) then the $\mathcal{F}$-coverable and $\mathcal{F}$-emulable families of graphs, are also minor ideals. There are not very many operators known. So for every $\mathcal{F}$, there is an analog of the notorious Fellows' conjecture — for what $\mathcal{F}$, does $\mathcal{F}$-coverable = $\mathcal{F}$-emulable? This works for

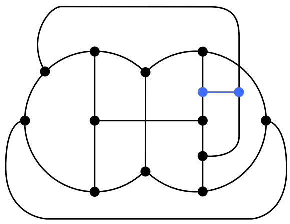

Figure 8.1: "Unification" of pictures of G_1 and G_2 using a nontrivial bridge.

$\mathcal{F}$ = outerplanar, and fails for $\mathcal{F}$=planar. In between are the disk dimension k classes: The *disk dimension* of a planar graph G is the least number k for which G embeds in the plane minus k open disks with every vertex on the boundary of some disk. It should be easy to show that for $\mathcal{F}$ = outerplanar, the situation is even stronger:

$$\mathcal{F}\text{-emulable } = \mathcal{F}\text{-coverable } = \mathcal{F}.$$

Is it possible that the same holds for $\mathcal{F}$ being any disk dimension k class of graphs?

In this respect, one can naturally think of extending the idea of emulations and coverings to other surfaces. What are the relations among those graph classes? We know that planar-emulable $\subseteq$ projective-planar-emulable and analogously for graph coverings. What is the relationship in the other direction? What happens for surfaces with higher genus? And how about bounded treewidth?

Understanding emulations and coverings for various graph classes could have an interesting impact from the algorithmic point of view. However, not being able to say much about the planar-emulable graphs, providing any sort of answer to the other questions seems as an ultimately difficult task.

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Appendix A

Lists of Generated Graphs

A.1 $\mathcal{C}_3$

- $\mathcal{C}_3^{[0]} = \mathcal{C}_3$
- $\mathcal{C}_3^{[1]} = \mathcal{C}_3^{[0]} {<} 8\left\{ {0,2,5 \atop 3,6,7} \right\}$,
  in [6] referred to as $\mathcal{C}_3^{[18]}$
- $\mathcal{C}_3^{[2]} = \mathcal{C}_3^{[0]} {<} 8\left\{ {0,3,6 \atop 2,5,7} \right\}$,
  in [6] referred to as $\mathcal{C}_3^{[19]}$
- $\mathcal{C}_3^{[3]} = \mathcal{C}_3^{[0]} + \{1,8\} + \{2,0\}$,
  in [6] referred to as $\mathcal{C}_3^{[1]}$
- $\mathcal{C}_3^{[4]} = \mathcal{C}_3^{[0]} \pentagon \{0,1,2,3,4\}$,
  in [6] referred to as $\mathcal{C}_3^{[2]}$
- $\mathcal{C}_3^{[5]} = \mathcal{C}_3^{[0]} {<} 2\left\{ {3,5 \atop 1,8} \right\} - \{3,5\} \simeq \mathcal{D}_2$
- $\mathcal{C}_3^{[6]} = \mathcal{C}_3^{[1]} + \{4,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[8]}$
- $\mathcal{C}_3^{[7]} = \mathcal{C}_3^{[1]} + \{1,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[4]}$
- $\mathcal{C}_3^{[8]} = \mathcal{C}_3^{[1]} + \{1,9\} + \{6,0\}$,
  in [6] referred to as $\mathcal{C}_3^{[6]}$
- $\mathcal{C}_3^{[9]} = \mathcal{C}_3^{[2]} + \{1,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[11]}$
- $\mathcal{C}_3^{[10]} = \mathcal{C}_3^{[2]} + \{1,9\} + \{2,0\}$,
  in [6] referred to as $\mathcal{C}_3^{[12]}$
- $\mathcal{C}_3^{[11]} = \mathcal{C}_3^{[3]} + \{4,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[3]}$
- $\mathcal{C}_3^{[12]} = \mathcal{C}_3^{[6]} + \{0,6\} + \{9,1\}$,
  in [6] referred to as $\mathcal{C}_3^{[5]}$
- $\mathcal{C}_3^{[13]} = \mathcal{C}_3^{[6]} + \{0,7\}$,
  in [6] referred to as $\mathcal{C}_3^{[21]}$
- $\mathcal{C}_3^{[14]} = \mathcal{C}_3^{[6]} + \{1,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[25]}$
- $\mathcal{C}_3^{[15]} = \mathcal{C}_3^{[8]} + \{4,9\}$,
  in [6] referred to as $\mathcal{C}_3^{[10]}$
- $\mathcal{C}_3^{[16]} = \mathcal{C}_3^{[9]} + \{0,7\} + \{9,4\}$,
  in [6] referred to as $\mathcal{C}_3^{[9]}$
- $\mathcal{C}_3^{[17]} = \mathcal{C}_3^{[9]} + \{4,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[31]}$
- $\mathcal{C}_3^{[18]} = \mathcal{C}_3^{[10]} + \{4,9\}$,
  in [6] referred to as $\mathcal{C}_3^{[13]}$
- $\mathcal{C}_3^{[19]} = \mathcal{C}_3^{[11]} + \{0,3\}$,
  in [6] referred to as $\mathcal{C}_3^{[14]}$
- $\mathcal{C}_3^{[20]} = \mathcal{C}_3^{[11]} + \{0,7\}$,
  in [6] referred to as $\mathcal{C}_3^{[20]}$
- $\mathcal{C}_3^{[21]} = \mathcal{C}_3^{[12]} + \{0,7\}$,
  in [6] referred to as $\mathcal{C}_3^{[15]}$

- $\mathcal{C}_3^{[22]} = \mathcal{C}_3^{[12]} + \{4,9\}$,
  in [6] referred to as $\mathcal{C}_3^{[23]}$,$\mathcal{C}_3^{[28]}$, $\mathcal{C}_3^{[34]}$ (the result was included multiple times)

- $\mathcal{C}_3^{[23]} = \mathcal{C}_3^{[12]} + \{0,3\}$,
  in [6] referred to as $\mathcal{C}_3^{[26]}$, $\mathcal{C}_3^{[33]}$ (the result was included multiple times)

- $\mathcal{C}_3^{[24]} = \mathcal{C}_3^{[13]} + \{1,8\}$,
  in [6] referred to as $\mathcal{C}_3^{[22]}$

- $\mathcal{C}_3^{[25]} = \mathcal{C}_3^{[13]} + \{1,9\}$,
  in [6] referred to as $\mathcal{C}_3^{[7]}$

- $\mathcal{C}_3^{[26]} = \mathcal{C}_3^{[15]} + \{3,7\}$,
  in [6] referred to as $\mathcal{C}_3^{[17]}$

- $\mathcal{C}_3^{[27]} = \mathcal{C}_3^{[16]} + \{0,6\}$,
  in [6] referred to as $\mathcal{C}_3^{[16]}$

- $\mathcal{C}_3^{[28]} = \mathcal{C}_3^{[16]} + \{1,9\}$,
  in [6] referred to as $\mathcal{C}_3^{[27]}$,$\mathcal{C}_3^{[30]}$ (the result was included multiple times)

- $\mathcal{C}_3^{[29]} = \mathcal{C}_3^{[18]} + \{0,7\}$,
  in [6] referred to as $\mathcal{C}_3^{[32]}$

- $\mathcal{C}_3^{[30]} = \mathcal{C}_3^{[24]} + \{1,9\}$,
  in [6] referred to as $\mathcal{C}_3^{[24]}$,$\mathcal{C}_3^{[29]}$ (the result was included multiple times)

A.2 $\mathcal{D}_2$

- $\mathcal{D}_2^{[0]} =$

- $\mathcal{D}_2^{[1]} = \mathcal{D}_2^{[0]} < 9\left\{\substack{1,3,5\\4,6,7}\right\}$,
  in [6] referred to as $\mathcal{D}_2^{[21]}$

- $\mathcal{D}_2^{[2]} = \mathcal{D}_2^{[0]} + \{0,9\} + \{6,1\}$,
  in [6] referred to as $\mathcal{D}_2^{[1]}$

- $\mathcal{D}_2^{[3]} = \mathcal{D}_2^{[0]\substack{+\{1,3\}+\{9,0\}+\{5,6\}\\+\{9,8\}+\{4,7\}+\{9,2\}}}$,
  in [6] referred to as $\mathcal{D}_2^{[11]}$

- $\mathcal{D}_2^{[4]} = \mathcal{D}_2^{[0]} < 3\left\{\substack{4,5\\0,9}\right\} - \{4,5\} \simeq \mathcal{E}_2$

- $\mathcal{D}_2^{[5]} = \mathcal{D}_2^{[0]} \pentagon \{1,0,3,4,2\}$,
  in [6] referred to as $\mathcal{D}_2^{[19]}$

- $\mathcal{D}_2^{[6]} = \mathcal{D}_2^{[0]} \hexagon \{0,2,8\}$,
  in [6] referred to as $\mathcal{D}_2^{[20]}$

- $\mathcal{D}_2^{[7]} = \mathcal{D}_2^{[1]} + \{0,10\} + \{6,1\}$,
  in [6] referred to as $\mathcal{D}_2^{[5]}$

- $\mathcal{D}_2^{[8]} = \mathcal{D}_2^{[1]\substack{+\{2,10\}+\{3,1\}+\{9,0\}\\+\{5,6\}+\{9,8\}+\{4,7\}}}$,
  in [6] referred to as $\mathcal{D}_2^{[3]}$

- $\mathcal{D}_2^{[9]} = \mathcal{D}_2^{[1]} + \{8,10\}$,
  in [6] referred to as $\mathcal{D}_2^{[9]}$

- $\mathcal{D}_2^{[10]} = \mathcal{D}_2^{[1]} + \{2,9\} + \{1,7\}$,
  in [6] referred to as $\mathcal{D}_2^{[29]}$

- $\mathcal{D}_2^{[11]} = \mathcal{D}_2^{[2]} + \{2,9\} + \{1,7\}$,
  in [6] referred to as $\mathcal{D}_2^{[15]}$

- $\mathcal{D}_2^{[12]} = \mathcal{D}_2^{[5]} + \{8,9\} + \{7,6\}$,
  in [6] referred to as $\mathcal{D}_2^{[22]}$

- $\mathcal{D}_2^{[13]} = \mathcal{D}_2^{[7]} + \{2, 10\}$,
  in [6] referred to as $\mathcal{D}_2^{[8]}$
- $\mathcal{D}_2^{[14]} = \mathcal{D}_2^{[7]} + \{2, 9\} + \{1, 7\}$,
  in [6] referred to as $\mathcal{D}_2^{[16]}$
- $\mathcal{D}_2^{[15]} = \mathcal{D}_2^{[9]} + \{1, 7\} + \{9, 2\}$,
  in [6] referred to as $\mathcal{D}_2^{[25]}$
- $\mathcal{D}_2^{[16]} = \mathcal{D}_2^{[10]} + \{0, 10\}$,
  in [6] referred to as $\mathcal{D}_2^{[6]}$
- $\mathcal{D}_2^{[17]} = \mathcal{D}_2^{[10]} + \{8, 9\} + \{5, 6\}$,
  in [6] referred to as $\mathcal{D}_2^{[18]}$
- $\mathcal{D}_2^{[18]} = \mathcal{D}_2^{[11]} + \{8, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[23]}$
- $\mathcal{D}_2^{[19]} = \mathcal{D}_2^{[12]} + \{0, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[24]}$
- $\mathcal{D}_2^{[20]} = \mathcal{D}_2^{[13]} + \{5, 7\} + \{9, 8\}$,
  in [6] referred to as $\mathcal{D}_2^{[34]}$,$\mathcal{D}_2^{[40]}$ (the result was included multiple times)
- $\mathcal{D}_2^{[21]} = \mathcal{D}_2^{[13]} + \{8, 10\}$,
  in [6] referred to as $\mathcal{D}_2^{[35]}$
- $\mathcal{D}_2^{[22]} = \mathcal{D}_2^{[14]} + \{8, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[4]}$
- $\mathcal{D}_2^{[23]} = \mathcal{D}_2^{[15]} + \{6, 7\}$,
  in [6] referred to as $\mathcal{D}_2^{[17]}$
- $\mathcal{D}_2^{[24]} = \mathcal{D}_2^{[15]} + \{0, 10\}$,
  in [6] referred to as $\mathcal{D}_2^{[33,36]}$
- $\mathcal{D}_2^{[25]} = \mathcal{D}_2^{[16]} + \{5, 6\} + \{9, 8\}$,
  in [6] referred to as $\mathcal{D}_2^{[27]}$
- $\mathcal{D}_2^{[26]} = \mathcal{D}_2^{[17]} + \{0, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[28]}$
- $\mathcal{D}_2^{[27]} = \mathcal{D}_2^{[18]} + \{6, 7\}$,
  in [6] referred to as $\mathcal{D}_2^{[37]}$
- $\mathcal{D}_2^{[28]} = \mathcal{D}_2^{[18]} + \{5, 6\}$,
  in [6] referred to as $\mathcal{D}_2^{[10]}$
- $\mathcal{D}_2^{[29]} = \mathcal{D}_2^{[19]} + \{2, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[39]}$
- $\mathcal{D}_2^{[30]} = \mathcal{D}_2^{[20]} + \{2, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[30,41]}$
- $\mathcal{D}_2^{[31]} = \mathcal{D}_2^{[20]} + \{4, 7\}$,
  in [6] referred to as $\mathcal{D}_2^{[38]}$
- $\mathcal{D}_2^{[32]} = \mathcal{D}_2^{[22]} + \{5, 7\}$,
  in [6] referred to as $\mathcal{D}_2^{[13]}$
- $\mathcal{D}_2^{[33]} = \mathcal{D}_2^{[22]} + \{5, 6\}$,
  in [6] referred to as $\mathcal{D}_2^{[2]}$
- $\mathcal{D}_2^{[34]} = \mathcal{D}_2^{[23]} + \{0, 10\}$,
  in [6] referred to as $\mathcal{D}_2^{[26]}$
- $\mathcal{D}_2^{[35]} = \mathcal{D}_2^{[23]} + \{0, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[7]}$
- $\mathcal{D}_2^{[36]} = \mathcal{D}_2^{[25]} + \{0, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[31]}$
- $\mathcal{D}_2^{[37]} = \mathcal{D}_2^{[26]} + \{1, 3\}$,
  in [6] referred to as $\mathcal{D}_2^{[14]}$
- $\mathcal{D}_2^{[38]} = \mathcal{D}_2^{[34]} + \{0, 9\}$,
  in [6] referred to as $\mathcal{D}_2^{[32]}$
- $\mathcal{D}_2^{[39]} = \mathcal{D}_2^{[34]} + \{3, 6\}$,
  in [6] referred to as $\mathcal{D}_2^{[12]}$

A.3 $K_{4,4}-e$

- $K_{4,4}-e^{[0]} = K_{4,4}-e$
- $K_{4,4}-e^{[1]} = K_{4,4}-e^{[0]}+\{0,4\}$, in[6] referred to as $K_{4,4}-e^{[1]}$
- $K_{4,4}-e^{[2]} = K_{4,4}-e^{[0]}<1\{{0,4 \atop 5,6}\}$, in[6] referred to as $K_{4,4}-e^{[7]}$
- $K_{4,4}-e^{[3]} = K_{4,4}-e^{[2]}+\{2,7\}$, in[6] referred to as $K_{4,4}-e^{[23]}$
- $K_{4,4}-e^{[4]} = K_{4,4}-e^{[2]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[22]}$
- $K_{4,4}-e^{[5]} = K_{4,4}-e^{[2]}<4\{{2,7 \atop 3,8}\}$, in[6] referred to as $K_{4,4}-e^{[45]}$
- $K_{4,4}-e^{[6]} = K_{4,4}-e^{[2]}<2\{{0,5 \atop 4,6}\}$, in[6] referred to as $K_{4,4}-e^{[44]}$
- $K_{4,4}-e^{[7]} = K_{4,4}-e^{[2]}+\{0,5\}$, in[6] referred to as $K_{4,4}-e^{[21]}$
- $K_{4,4}-e^{[8]} = K_{4,4}-e^{[2]}+\{2,8\}+\{1,0\}$, in[6] referred to as $K_{4,4}-e^{[3]}$
- $K_{4,4}-e^{[9]} = K_{4,4}-e^{[8]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[11]}$
- $K_{4,4}-e^{[10]} = K_{4,4}-e^{[8]}+\{3,7\}$, in[6] referred to as $K_{4,4}-e^{[14]}$
- $K_{4,4}-e^{[11]} = K_{4,4}-e^{[8]}+\{2,3\}$, in[6] referred to as $K_{4,4}-e^{[32]}$
- $K_{4,4}-e^{[12]} = K_{4,4}-e^{[8]}+\{0,5\}$, in[6] referred to as $K_{4,4}-e^{[29]}$
- $K_{4,4}-e^{[13]} = K_{4,4}-e^{[12]}+\{3,7\}$, in[6] referred to as $K_{4,4}-e^{[52]}$
- $K_{4,4}-e^{[14]} = K_{4,4}-e^{[13]}+\{5,6\}$, in[6] referred to as $K_{4,4}-e^{[56]}$
- $K_{4,4}-e^{[15]} = K_{4,4}-e^{[11]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[57]}$ and $K_{4,4}-e^{[59]}$
- $K_{4,4}-e^{[16]} = K_{4,4}-e^{[11]}+\{3,7\}$, in[6] referred to as $K_{4,4}-e^{[54]}$
- $K_{4,4}-e^{[17]} = K_{4,4}-e^{[16]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[76]}$
- $K_{4,4}-e^{[18]} = K_{4,4}-e^{[16]}+\{5,6\}$, in[6] referred to as $K_{4,4}-e^{[67]}$
- $K_{4,4}-e^{[19]} = K_{4,4}-e^{[15]}+\{4,5\}$, in[6] referred to as $K_{4,4}-e^{[69]}$
- $K_{4,4}-e^{[20]} = K_{4,4}-e^{[10]}+\{5,6\}$, in[6] referred to as $K_{4,4}-e^{[28]}$

- $K_{4,4}-e^{[21]} = K_{4,4}-e^{[10]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[48]}$
- $K_{4,4}-e^{[22]} = K_{4,4}-e^{[9]}+\{4,5\}$, in[6] referred to as $K_{4,4}-e^{[25]}$
- $K_{4,4}-e^{[23]} = K_{4,4}-e^{[7]}<4\{{2,7 \atop 3,8}\}$, in[6] referred to as $K_{4,4}-e^{[65]}$
- $K_{4,4}-e^{[24]} = K_{4,4}-e^{[7]}+\{2,7\}$, in[6] referred to as $K_{4,4}-e^{[53]}$
- $K_{4,4}-e^{[25]} = K_{4,4}-e^{[7]}+\{2,8\}$, in[6] referred to as $K_{4,4}-e^{[4]}$
- $K_{4,4}-e^{[26]} = K_{4,4}-e^{[7]}<5\{{0,2,3 \atop 1,7}\}$, in[6] referred to as $K_{4,4}-e^{[64]}$
- $K_{4,4}-e^{[27]} = K_{4,4}-e^{[7]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[51]}$
- $K_{4,4}-e^{[28]} = K_{4,4}-e^{[27]}+\{2,8\}$, in[6] referred to as $K_{4,4}-e^{[12]}$
- $K_{4,4}-e^{[29]} = K_{4,4}-e^{[27]}<0\{{2,5 \atop 3,8}\}-\{2,5\}$, in[6] referred to as $K_{4,4}-e^{[66]}$
- $K_{4,4}-e^{[30]} = K_{4,4}-e^{[27]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[58]}$
- $K_{4,4}-e^{[31]} = K_{4,4}-e^{[30]}+\{2,8\}$, in[6] referred to as $K_{4,4}-e^{[26]}$
- $K_{4,4}-e^{[32]} = K_{4,4}-e^{[31]}+\{1,4\}$, in[6] referred to as $K_{4,4}-e^{[33]}$
- $K_{4,4}-e^{[33]} = K_{4,4}-e^{[31]}<1\{{5,7 \atop 6,8}\}-\{5,7\}-\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[39]}$
- $K_{4,4}-e^{[34]} = K_{4,4}-e^{[33]}+\{1,4\}$, in[6] referred to as $K_{4,4}-e^{[19]}$
- $K_{4,4}-e^{[35]} = K_{4,4}-e^{[33]}<5\{{0,3 \atop 2,9}\}-\{0,3\}-\{2,8\}$, in[6] referred to as $K_{4,4}-e^{[75]}$
- $K_{4,4}-e^{[36]} = K_{4,4}-e^{[33]}+\{2,9\}$, in[6] referred to as $K_{4,4}-e^{[40]}$
- $K_{4,4}-e^{[37]} = K_{4,4}-e^{[36]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[55]}$
- $K_{4,4}-e^{[38]} = K_{4,4}-e^{[36]}+\{1,4\}$, in[6] referred to as $K_{4,4}-e^{[6]}$
- $K_{4,4}-e^{[39]} = K_{4,4}-e^{[38]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[16]}$
- $K_{4,4}-e^{[40]} = K_{4,4}-e^{[39]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[42]}$
- $K_{4,4}-e^{[41]} = K_{4,4}-e^{[37]}+\{6,8\}$, in[6] referred to as $K_{4,4}-e^{[27]}$
- $K_{4,4}-e^{[42]} = K_{4,4}-e^{[37]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[68]}$

- $K_{4,4}–e^{[43]} = K_{4,4}–e^{[42]}+\{6,8\}$, in[6] referred to as $K_{4,4}–e^{[70]}$ and $K_{4,4}–e^{[74]}$ (the result was included multiple times)
- $K_{4,4}–e^{[44]} = K_{4,4}–e^{[35]}+\{0,3\}+\{2,10\}+\{7,5\}+\{6,9\}+\{4,1\}+\{2,8\}$, in[6] referred to as $K_{4,4}-e^{[35]}$
- $K_{4,4}-e^{[45]} = K_{4,4}-e^{[44]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[73]}$
- $K_{4,4}-e^{[46]} = K_{4,4}-e^{[44]}$
- $K_{4,4}-e^{[47]} = K_{4,4}-e^{[45]}$
- $K_{4,4}-e^{[48]} = K_{4,4}-e^{[34]}+\{5,7\}+\{6,9\}$, in[6] referred to as $K_{4,4}-e^{[37]}$
- $K_{4,4}-e^{[49]} = K_{4,4}-e^{[48]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[62]}$
- $K_{4,4}–e^{[50]} = K_{4,4}–e^{[29]}+\{6,8\}$, in[6] referred to as $K_{4,4}–e^{[38]}$ and $K_{4,4}–e^{[77]}$ (the result was included multiple times)
- $K_{4,4}-e^{[51]} = K_{4,4}-e^{[29]}+\{3,9\}+\{2,0\}$, in[6] referred to as $K_{4,4}-e^{[20]}$
- $K_{4,4}-e^{[52]} = K_{4,4}-e^{[51]}+\{6,8\}$, in[6] referred to as $K_{4,4}-e^{[5]}$
- $K_{4,4}-e^{[53]} = K_{4,4}-e^{[51]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[30]}$
- $K_{4,4}–e^{[54]} = K_{4,4}–e^{[53]}+\{6,8\}$, in[6] referred to as $K_{4,4}–e^{[41]}$ amd $K_{4,4}–e^{[71]}$ (the result was included multiple times)
- $K_{4,4}–e^{[55]} = K_{4,4}–e^{[54]}+\{0,4\}$, in[6] referred to as $K_{4,4}–e^{[61]}$ and $K_{4,4}–e^{[72]}$ (the result was included multiple times)
- $K_{4,4}-e^{[56]} = K_{4,4}-e^{[52]}+\{0,4\}$, in[6] referred to as $K_{4,4}-e^{[17]}$
- $K_{4,4}–e^{[57]} = K_{4,4}–e^{[50]}+\{0,4\}$, in[6] referred to as $K_{4,4}–e^{[60]}$ and $K_{4,4}–e^{[63]}$ (the result was included multiple times)
- $K_{4,4}-e^{[58]} = K_{4,4}-e^{[58]}+\{3,9\}$, in[6] referred to as $K_{4,4}-e^{[34]}$
- $K_{4,4}-e^{[59]} = K_{4,4}-e^{[28]}+\{1,4\}$, in[6] referred to as $K_{4,4}-e^{[50]}$
- $K_{4,4}–e^{[60]} = K_{4,4}–e^{[26]}+\{2,5\}$, in[6] referred to as $K_{4,4}–e^{[36]}$ and $K_{4,4}–e^{[43]}$ (the result was included multiple times)

- $K_{4,4}-e^{[61]} = K_{4,4}-e^{[25]}+\{3,7\}$, in[6] referred to as $K_{4,4}-e^{[15]}$
- $K_{4,4}-e^{[62]} = K_{4,4}-e^{[25]}+\{1,4\}$, in[6] referred to as $K_{4,4}-e^{[18]}$
- $K_{4,4}-e^{[63]} = K_{4,4}-e^{[4]}+\{2,7\}$, in[6] referred to as $K_{4,4}-e^{[49]}$
- $K_{4,4}-e^{[64]} = K_{4,4}-e^{[4]}+\{4,5\}$, in[6] referred to as $K_{4,4}-e^{[31]}$
- $K_{4,4}-e^{[65]} = K_{4,4}-e^{[1]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[9]}$
- $K_{4,4}-e^{[66]} = K_{4,4}-e^{[1]}+\{1,2\}$, in[6] referred to as $K_{4,4}-e^{[2]}$
- $K_{4,4}-e^{[67]} = K_{4,4}-e^{[1]}+\{0,5\}$, in[6] referred to as $K_{4,4}-e^{[8]}$
- $K_{4,4}-e^{[68]} = K_{4,4}-e^{[67]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{46[]}$
- $K_{4,4}-e^{[69]} = K_{4,4}-e^{[68]}+\{4,6\}$, in[6] referred to as $K_{4,4}-e^{[47]}$
- $K_{4,4}-e^{[70]} = K_{4,4}-e^{[66]}+\{1,7\}$, in[6] referred to as $K_{4,4}-e^{[10]}$
- $K_{4,4}-e^{[71]} = K_{4,4}-e^{[66]}+\{3,7\}$, in[6] referred to as $K_{4,4}-e^{[13]}$
- $K_{4,4}-e^{[72]} = K_{4,4}-e^{[71]}+\{4,5\}$, in[6] referred to as $K_{4,4}-e^{[24]}$

A.4 Violating splits of $K_{4,4}-e$

There are two graphs with $K_{4,4}$–e minor for which a violating split exists — the graphs $K_{4,4}$–$e^{[44]}$ and $K_{4,4}$–$e^{[45]}$. In both cases, they lead to a subdivision of a graph isomorphic to $K_{4,4}$–$e^{[35]}$. The particular violating splits are listed below.

- $K_{4,4}-e^{[44]} < 0\{{}^{2,8}_{3,10}\}-\{2,8\}-\{3,10\}-\{1,4\}-\{2,5\}-\{6,9\}-\{7,9\}$
- $K_{4,4}-e^{[44]} < 0\{{}^{2,8}_{3,10}\}-\{2,8\}-\{3,10\}$
- $K_{4,4}-e^{[44]} < 1\{{}^{4,8}_{6,9}\}-\{4,8\}-\{6,9\}-\{0,2\}-\{3,10\}$
- $K_{4,4}-e^{[44]} < 3\{{}^{4,6}_{0,10}\}-\{0,10\}-\{2,5\}-\{2,8\}$
- $K_{4,4}-e^{[44]} < 3\{{}^{4,6}_{0,10}\}-\{0,10\}-\{2,5\}-\{2,8\}-\{7,9\}-\{1,4\}-\{1,6\}$
- $K_{4,4}-e^{[44]} < 5\{{}^{7,9}_{2,10}\}-\{2,10\}-\{7,9\}-\{0,3\}-\{1,6\}-\{2,8\}$

- $K_{4,4} - e^{[45]} < 5\left\{{7,9 \atop 2,10}\right\} - \{2,10\} - \{7,9\} - \{0,3\} - \{1,6\} - \{2,8\} - \{4,8\}$
- $K_{4,4} - e^{[45]} < 7\left\{{4,6 \atop 5,9}\right\} - \{4,6\} - \{5,9\} - \{1,6\} - \{2,10\} - \{4,8\} - \{0,2\} - \{0,3\}$
- $K_{4,4} - e^{[45]} < 7\left\{{4,6 \atop 5,9}\right\} - \{5,9\} - \{1,6\} - \{2,10\} - \{4,8\} - \{0,2\} - \{0,3\}$
- $K_{4,4} - e^{[45]} < 8\left\{{0,2 \atop 1,4}\right\} - \{0,2\} - \{1,4\} - \{3,10\} - \{6,9\} - \{2,5\} - \{5,7\}$
- $K_{4,4} - e^{[45]} < 9\left\{{1,6 \atop 5,7}\right\} - \{1,6\} - \{4,8\} - \{5,7\} - \{0,2\} - \{2,10\} - \{3,10\}$
- $K_{4,4} - e^{[45]} < 9\left\{{1,6 \atop 5,7}\right\} - \{1,6\} - \{4,8\} - \{5,7\} - \{0,2\} - \{2,10\} - \{3,10\} - \{4,6\}$