Taking and Breaking Games

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Motivation

We aim to attack several open problems of **combinatorial game theory**. This field of combinatorics studies sequential two-player games with no hidden information and no chance elements. The games in this area are usually computationally intractable, which makes them a great candidates to challenge our mathematical methods.

We focus here on combinatorial games involving heaps of tokens where players alternately choose a heap, remove some tokens and split the remaining heap into several other heaps according to given particular rules, a.k.a. **Taking and Breaking Games**.

Our goal is to answer the following typical questions about these games:

- *Which player wins, if both play optimally?*
- *What should be the winning strategy?*
- *What is the complexity class of these problems?*

Many such questions about this category of games are still open. They have been appearing on the top of the famous lists of unsolved problems since 1996. Among others in an ongoing series of papers published by the Cambridge University *Games of No Chance*.

Method

The main difficulty when analysing these games lies in the exponentially growing size of the game tree. Therefore the goal is to simplify the description of particular positions in the game in order to make them tractable. One of the techniques to achieve this is to use **Conway's Disjunctive Theory**. In this approach we perceive each pile as independent game and the composition of these "base" games we call a *sum of games*. Then we take advantage of the fact that each move in this game is another sum of games.

Now, in order to be able to reason about the uncountable number of positions in these games, we design a new representation of game positions that are succinct enough, so that they optimally contain only the necessary information and ignore everything else. In particular, we search for a *game monoid* $(M, \oplus)$ where $\oplus$ represents the sum of games. Furthermore, we look for a **game mapping function** Φ that maps each single-heap position of the game to an element in this monoid. If the monoid is designed in the way, that it is easy to tell the outcome of any corresponding position to each its value and if the computation of the function Φ is tractable, we have solved the game.

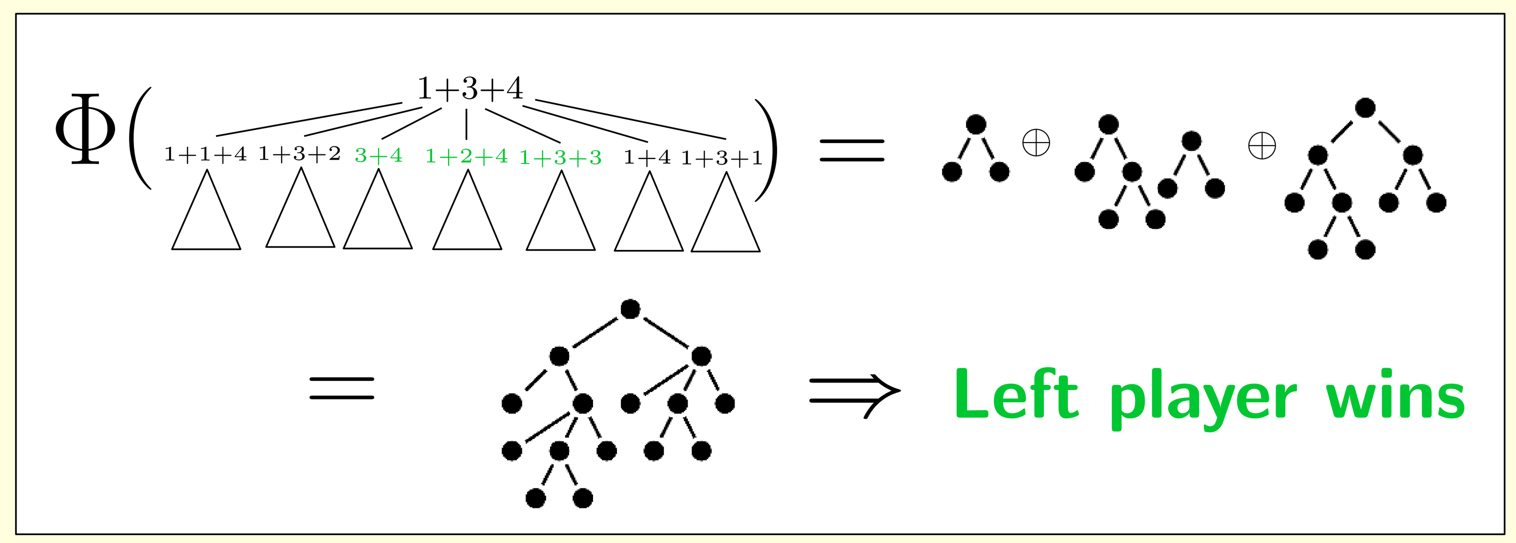
This approach proves to be useful in particular for games where the set of moves are for both players the same – the *impartial* games. In order to find such monoids for some particular families of these games, we applied a **computer-assisted** approach. We designed several game solving algorithms that allowed us to observe the behaviour of smaller games. Then we observe and prove patterns and properties which would allow us to design a suitable monoid and a tractable game mapping function.

$(\mathbb{N}, +)$ $\xmapsto[\text{(efficient)}]{\Phi}$ $(\mathbf{M}, \oplus)$

Set of game positions
Uncountable
Hard to compute the outcome

Game monoid
Countable/finite
Easy to compute the outcome

$$\Phi(1+3+4) = \Phi(1) \oplus \Phi(3) \oplus \Phi(4)$$



Example: Solving a position $1+3+4$ in *subtraction game* $\{1, 2 \mid 1, 3\}$.

Impartial Subtraction Games

In subtraction games players are allowed only to remove the tokens from the heaps. Each such game is described by *subtraction sets* of integers S, where player can subtract s tokens only if $s \in S$. The **Sprage-Grundy Theory** tells us that such impartial games can be mapped into a monoid on integers. Thus the task of solving a subtraction game lies in understanding the integral sequence of mapped single-heap values.

For example, we have studied the games with the following subtraction sets:

- $\{1 \cdot \ell, 2 \cdot \ell, \dots, k \cdot \ell\}$ with $k, \ell \in \mathbb{N}$,
- $\{1, b^a, b^{k_1 \cdot a}, b^{k_2 \cdot a}, \dots\}$ with $a, b \in \mathbb{N}$ and $k_i \in \mathbb{N}$ for all $i \in \mathbb{N}$,
- Infinite subtraction sets: primes, squares, Fibonacci numbers, etc.
- Bipartite games, where game-value sequence ultimately alternate $0, 1, 0, \dots$.

Furthermore, we made an attempt to answer the following question:

Given two subtraction games, are they essentially the same in terms of the game mapping function?

We have designed a theory of **canonical subtraction games**, which gives us the necessary and sufficient conditions to answer such questions. We have also provided an example of a subtraction game for which the outcome sequence is periodic and the game-value sequence is aperiodic.

Partisan Subtraction Games

The partisan generalisation of these games considers game-rules, where each player is restricted by a different subtraction set. By $\{a_1, a_2, \dots \mid b_1, b_2, \dots\}$ we denote a game where first player can subtract only a_i tokens while the second player only b_i for $i \in \mathbb{N}$. We have studied the following games:

- $\{1 \mid a\}$ with $a \in \mathbb{N}$,
- $\{1, 2 \mid 1, 3\}$ conjectured by Plumbeck in 1995,
- $\{1 \mid 1, 2, b_1, b_2, \dots\}$ where $b_i \in \mathbb{N}$ for all $i \in \mathbb{N}$,
- So called *Chessboard* subtraction games, where the first (second) player can create only heaps of odd (even) size, respectively.

Pure Breaking Games

Similar restriction as for subtraction games, when we allow the players only to split the heaps, gives us a class of games that has been (until now) studied only under impartial setting. Similarly we describe such games with *breaking sets* of integers which tell us into how many sub-heaps are players allowed to split a single heap. In this class we have studied the following games:

- $\{1 \mid 2, 3\}$, $\{1 \mid a)$ with $a \in \mathbb{N}$,
- $\{1, a_1, a_2, \dots \mid 1, b_1, b_2, \dots\}$ with $a_i, b_i \in \mathbb{N}$ are *odd* for all $i \in \mathbb{N}$,
- $\{1, 2, 3, a_1, a_2, \dots \mid 1, 2, 3, b_1, b_2, \dots\}$ with $a_i, b_i \in \mathbb{N}$ for all $i \in \mathbb{N}$,
- $\{c, a_1, a_2, \dots \mid c, b_1, b_2, \dots\}$ with $c > 1$ and $a_i, b_i > c$ for all $i \in \mathbb{N}$,
- $\{1 \mid 1, 2k, b_1, b_2, \dots\}$ with $k \geq 1$ and $b_i > 2k$ for all $i \in \mathbb{N}$,

We have also designed a **periodicity theorem** which enables us to computationally solve a large class of these games.

Complexity & Algorithms

Unlike for classical decision problems, where the complexity of most of the problems is known, for most combinatorial games we do not know what is the complexity of the question about the outcome of the game. We have shown, that a generalized subtraction game, where the input is a position in the game, a subtraction set, and a succinct representation of a set of won positions, is **PSPACE-hard**. Furthermore, we have shown that a partisan version of such game is **EXPTIME-hard**.

We have also designed several game solving algorithms that helped us with the theoretical results mentioned above. For instance we can compute the outcome of any standard Taking and Breaking game in time $\mathcal{O}(tkN^2)$, where t is maximal number of heaps to be split into, k is maximal number subtracted and N is maximal size of created heap. This is a significant improvement over the classical approach which runs in time $\mathcal{O}(kN^t)$.