Reliable characterization of the existence of solutions of parametric systems of equations

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Introduction

Systems of equations that occur in engineering practice often involve uncertain parameters. It is then often desirable to characterize the set of parameter values, for which the given equation system has a solution.

The main goal of the thesis is to develop and implement an algorithm that can perform such characterization. It should be based on floating point arithmetic and robust with respect to rounding errors.

Theoretical approach

Our approach is based on working with boxes (i.e. cartesian products of intervals) and performing computations via interval arithmetic to ensure the robustness with respect to rounding errors.

To perform a solution existence test for a given parametric system of equations we employ a method involving the use of the properties of Brouwer's topological degree [1].

Informally, the method is based on examining the behavior of the equation system on the boundary ∂X of its domain X and subsequently splitting ∂X into subsets, such that at least one component of the system is guaranteed to be non-zero on each of those subsets. We then make use of the algorithm presented in [2] that uses such information to compute the topological degree.

Main algorithm

We designed and implemented an algorithm characterizing sets of points $p \in P$, with respect to the solution existence of a given parametric system of equations $f_P(x, p) = 0$, where $f_P\colon X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function formed by arithmetic expressions involving constants, variables, parameters and any functions for which their interval enclosure can be computed via interval arithmetic [3].

The algorithm recursively splits the input parameter box P into sub-boxes in order to classify them either as *red* or *green*. For each parameter value in a green box, it is guaranteed that a solution of the given equation system exists, while for each parameter value in a red box, it is assured that the equation system has no solution.

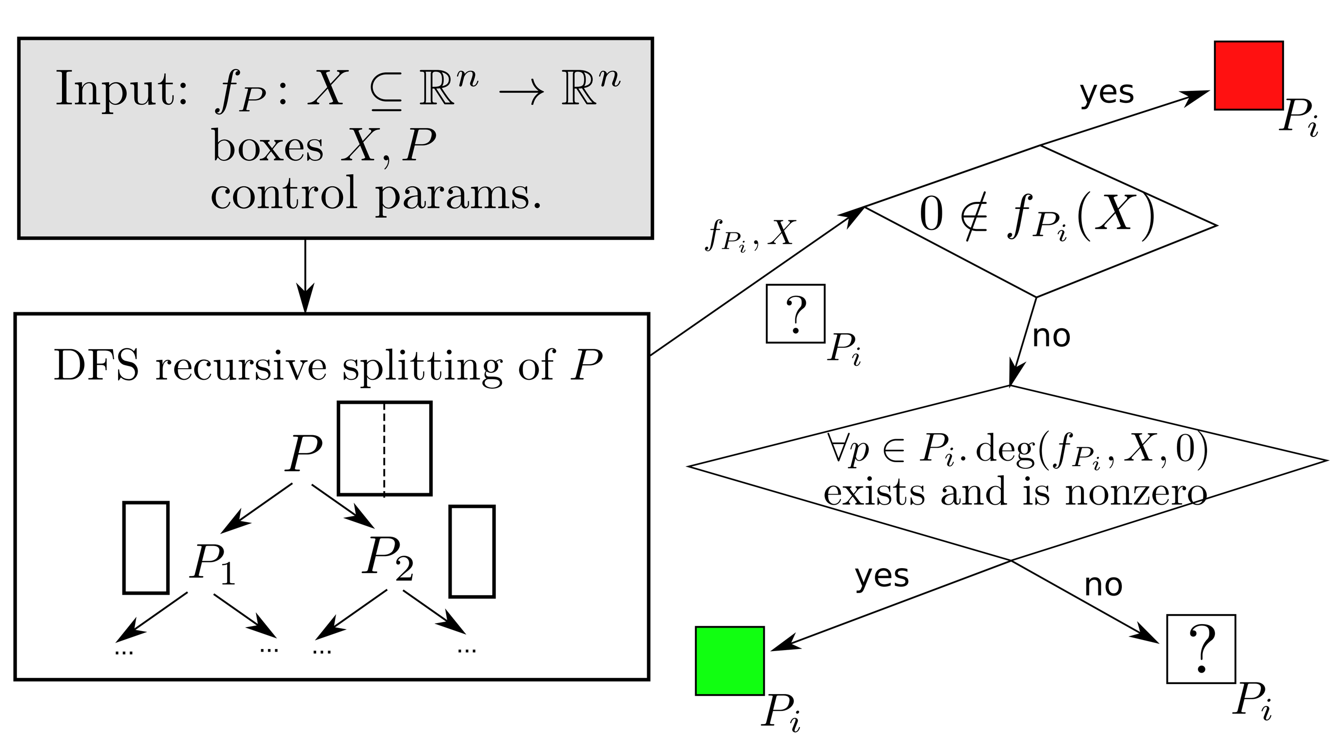

Figure 1: Top-level flow of the main algorithm.

Main algorithm (cont.)

The most of the logic is centered around the way the variable domain box X is manipulated during the computation. We discovered, that this greatly influences the ability to correctly localize a solution of the equation system. We devised, implemented and experimentally compared several of such ways and kept the design and implementation extensible enough for potentially new ways to be added in the future.

Each run of the algorithm can be highly parametrized and in the case of 1-D and 2-D equation systems, our implementation allows the result to be visualized.

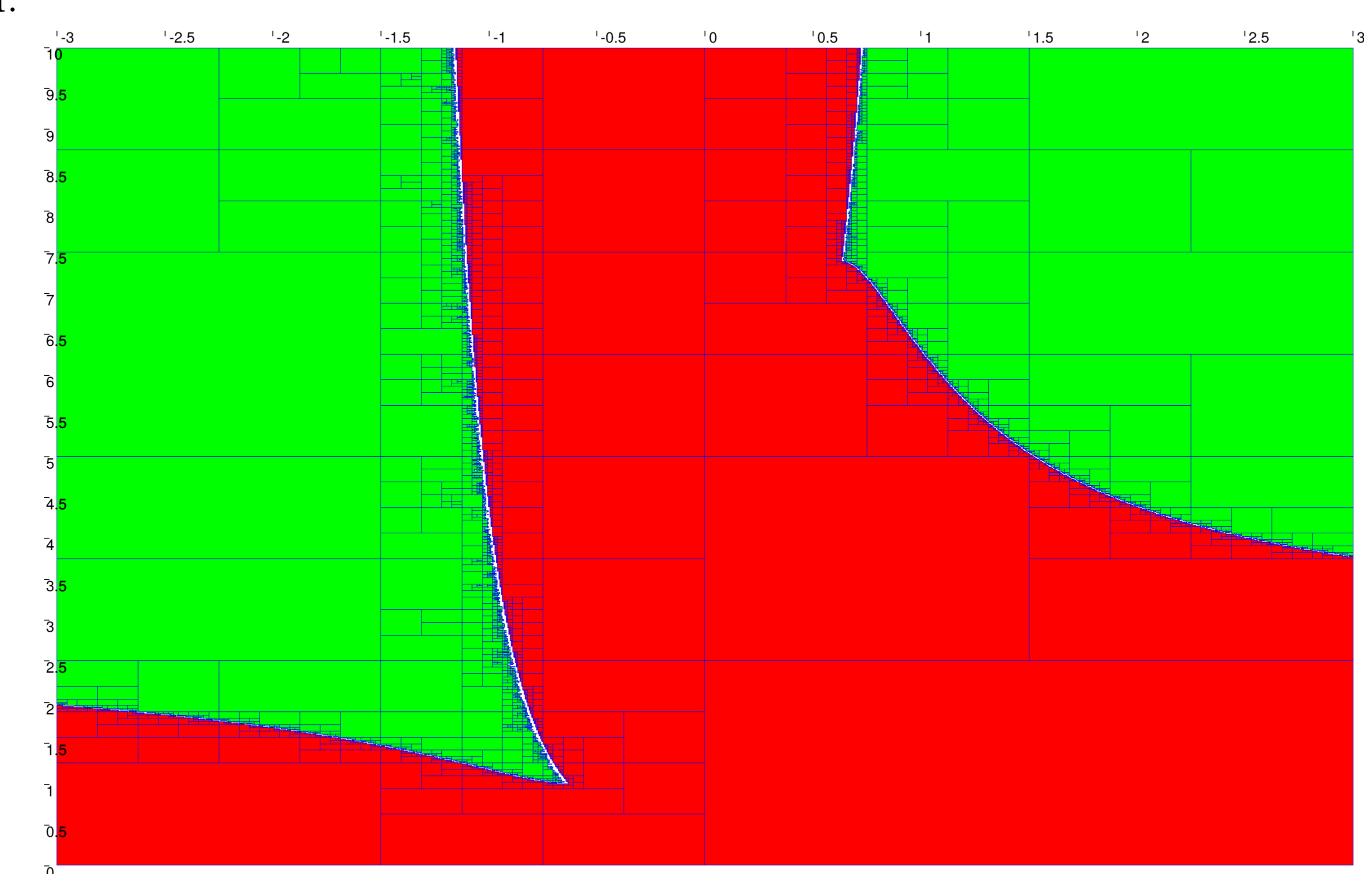

Figure 2: Visualized result for the system of two equations: $\arccos(px+q) - \frac{1}{2}$, $\sin(py+q) - \frac{1}{2}$, with variables $x, y \in [-2,2] \times [-2,2]$ and parameters $p, q \in [-1,1] \times [-1,1]$

Notable contributions

Apart from the main algorithm, we provided a custom implementation of the algorithm [2] for computing the topological degree. Our implementation proved to be noticeably faster than the original on experimental data (more than ten times on average).

We also managed to tightly and seamlessly integrate the degree computation into our main algorithm, that works with parametrized functions. To our knowledge, our approach is more efficient compared to the one suggested in [4].

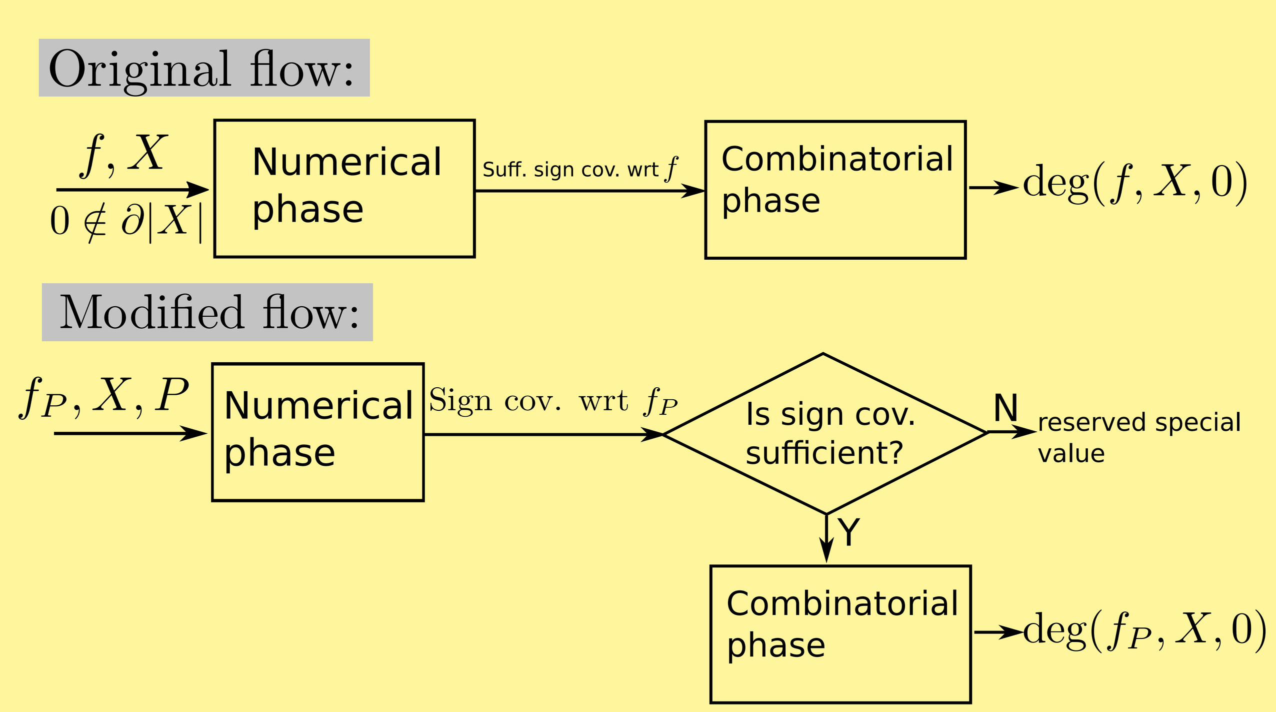

Figure 3: Differences in the flow of the algorithm [2] and our customization.

Experimental evaluation

For our main algorithm, we prepared a test suite consisting of representational classes of equation systems like systems of polynomial equations, trigonometric equations or equations formed by intersections of ellipsoids and hyper-surfaces.

We compared the quality of the different parametrizations of our algorithm with respect to the way the input variable domain box is internally manipulated. The quality was measured as the total area of the parameter space that the algorithm was able to classify into a red or a green sub-box.

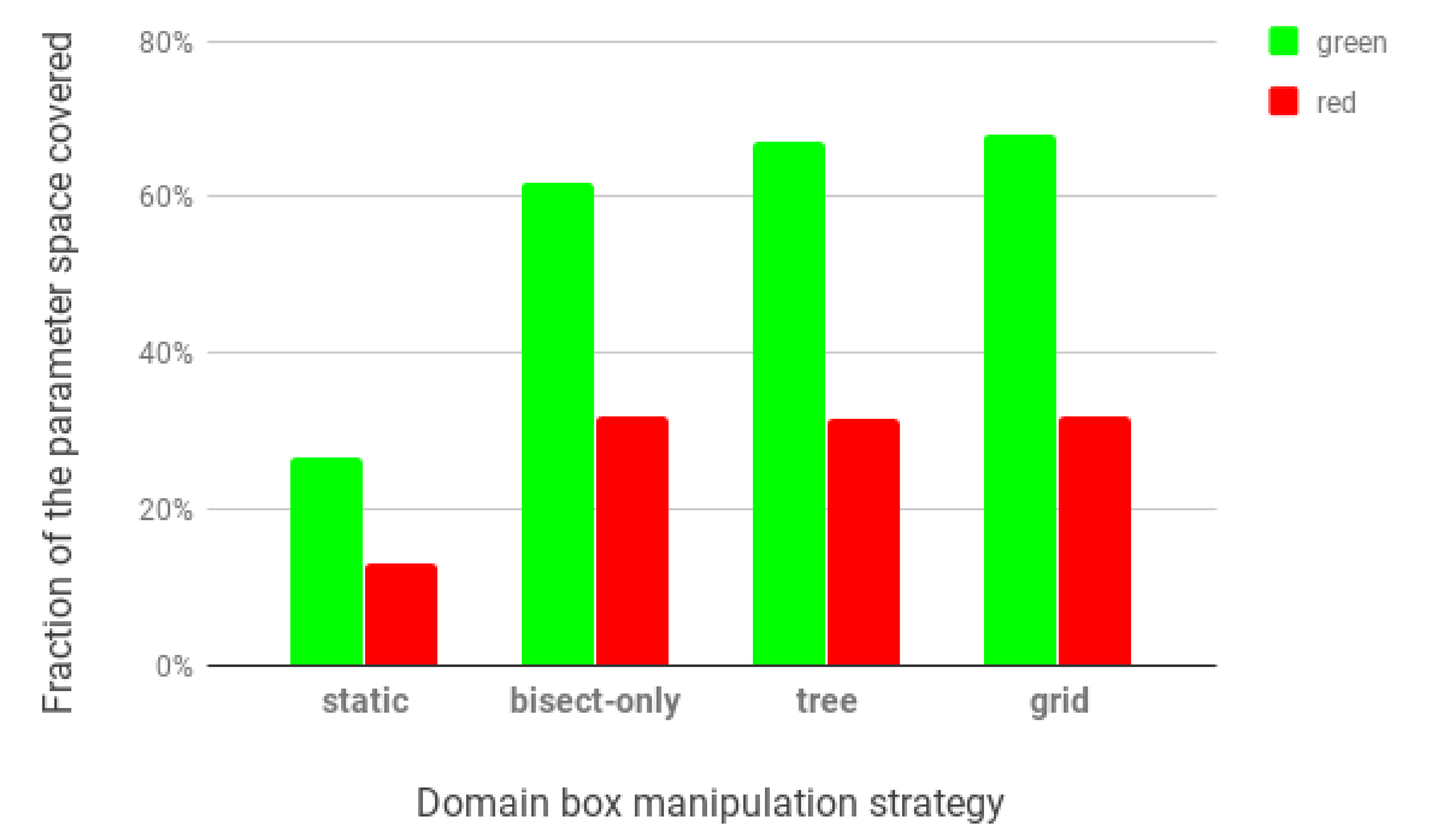

Figure 4: Quality of our algorithm for a sample system of equations formed by intersections of ellipsoids and hyper-surfaces.

References

[1] O'Regan, D.; Cho, Y. J.; et al. *Topological Degree Theory and Applications*. Boca Raton, FL: Chapman & Hall/CRC, 2006, ISBN 978-1584886488.

[2] Franek, P.; Ratschan, S. Effective Topological Degree Computation Based on Interval Arithmetic. *CoRR*, volume abs/1207.6331, 2012. Available from: `http://arxiv.org/abs/1207.6331`

[3] Jaulin, L.; Kieffer, M.; et al. Interval Analysis. In *Applied Interval Analysis: With Examples in Parameter and State Estimation, Robust Control and Robotics*, London: Springer London, 2001, ISBN 978-1-4471-0249-6, pp. 11–43, doi:10.1007/978-1-4471-0249-6_2. Available from: `https://doi.org/10.1007/978-1-4471-0249-6_2`

[4] Franek, P.; Ratschan, S.; et al. Quasi-decidability of a Fragment of the First-Order Theory of Real Numbers. *Journal of Automated Reasoning*, volume 57, no. 2, Aug 2016: pp. 157–185, ISSN 1573-0670, doi:10.1007/s10817-015-9351-3. Available from: `https://doi.org/10.1007/s10817-015-9351-3`