Methods of Efficient State Space Search for the Nurse Rostering Problem Using Branch-and-Price Approach

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Objectives

- Introduce an exact algorithm for general Nurse Rostering Problem with soft constraints.
- Application of Branch-and-Price decomposition method.
- Identify parts of algorithm that are suitable for improving by machine learning algorithms.
- Solve real-world instances up to the optimality.

Nurse Rostering Problem

Assigning employees to shifts in workplaces is a problem that is being solved every day on the entire planet. Usually, every place that is running operations involving larger number of people creates a working schedule for them, typically a few weeks onwards. Such working schedule specifies what the duty of an employee is for any given day.

We are given a planning horizon of length n, a set of employees $\mathcal{E}$, a set of skill competencies $\mathcal{J}$ and a set of a shift types $\mathcal{K}$. For each employee $i \in \mathcal{E}$ we have a set of patterns $\mathcal{P}_i$ which defines both desirable and undesirable shift sequences. Moreover, we are given a requirement for the preferred coverage R_{mjk} associated with the day m, served by employees with the skill competency j on the shift k for each day of the planning horizon. It is linked with the penalty c^u_{mjk} for understaffing and c^o_{mjk} for overstaffing it.

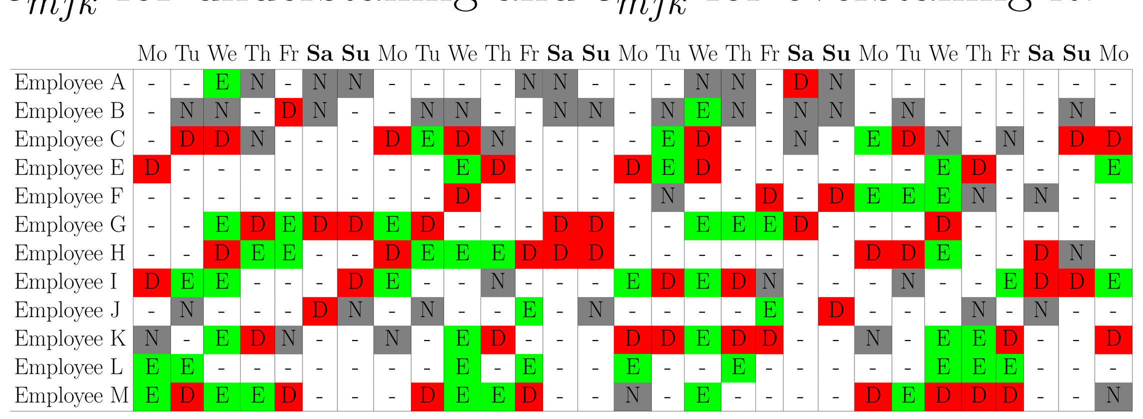

Figure 1: Optimal solution for *Motol-1* instance.

The objective is to minimize the sum of constraint violations (patterns) for each employee plus sum of violations of their shift requests plus sum of violations of coverage constraints. This problem is contained in $\mathcal{NP}$-hard complexity class.

Branch-and-Price method

Firstly, *original formulation* is decomposed into a *master problem* and *pricing problem*. Informally, master problem is a linear program that consists of variables related to all feasible states of a state space given implicitly by the original formulation.

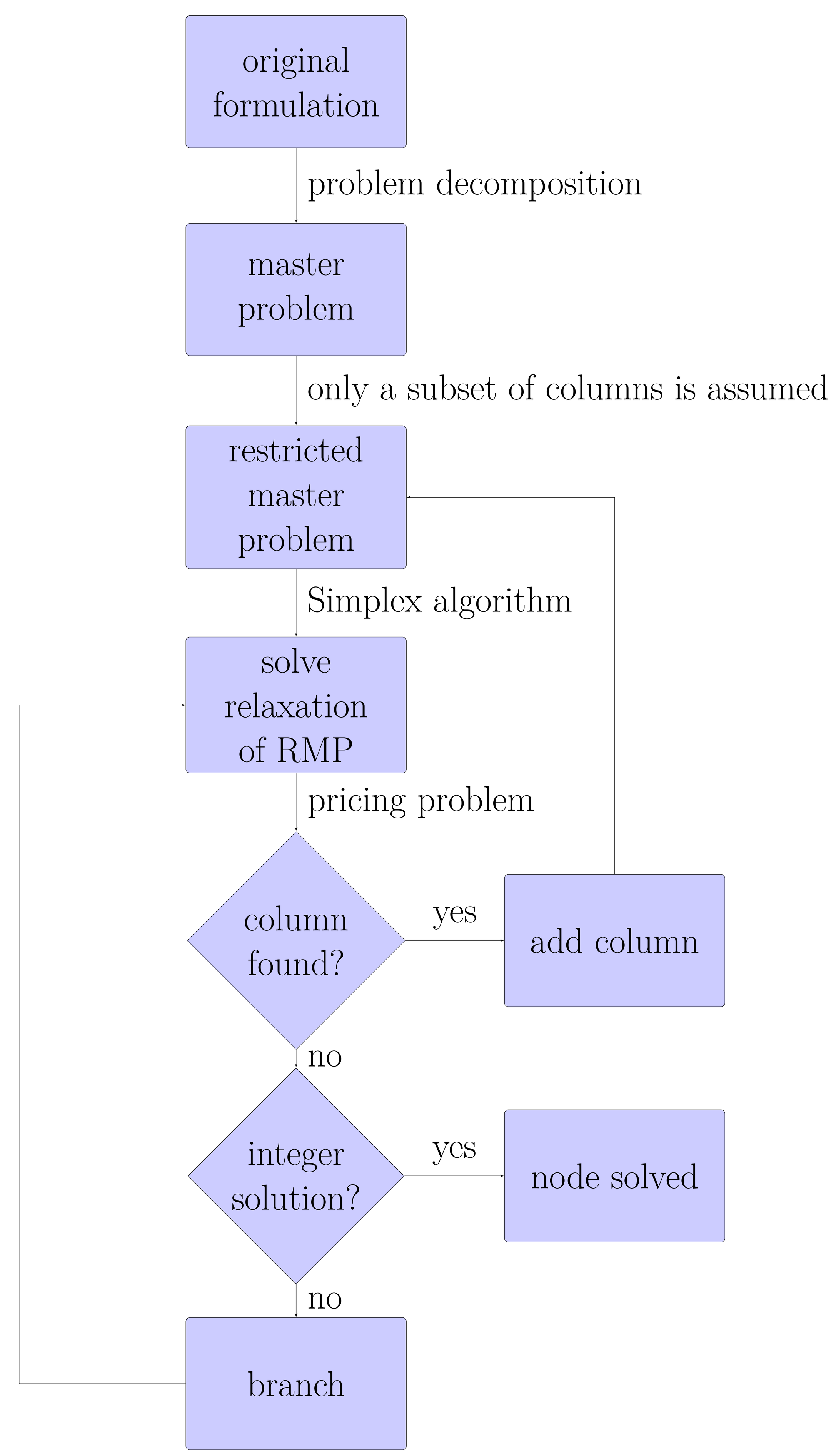

Figure 2: Block diagram of the Branch-and-Price method.

Master problem is solved using so-called *pricing problem* which works as a separation oracle in dual space. Unfortunately, its query complexity is still $\mathcal{NP}$-hard and it is called many times within the single solution of a problem instance.

Pricing Problem

Pricing problem provides new columns which are candidates for entering the Simplex basis and, thus, decreasing the primal objective value. Pricing problem is given by the following mathematical programming model

$$\min_{\mathbf{x}} -\gamma_i - \sum_{m\in\mathcal{M}} \sum_{k\in\mathcal{K}} \pi_{mk} x_{mk} + c_{il}(\mathbf{x}) \quad (1)$$

subject to

$$\forall m \in \mathcal{M} : \sum_{k\in\mathcal{K}} x_{mk} = 1 \quad (2)$$

$$\mathbf{x} \in \{0,1\}^{|\mathcal{M}|\times|\mathcal{K}|} \quad (3)$$

where $\mathbf{x}$ are *original variables* (assigning days to the shifts), γ_i and π_{mk} are constants in $\mathbb{R}$ and c_{il} is defined as

$$c_{il}(\mathbf{x}) = \sum_{p\in\mathcal{P}_i} c_p^{lb} \max\{0, lb_p - \#p\} + \\ + c_p^{ub} \max\{0, \#p - ub_p\} + \sum_{m\in\mathcal{M}} \sum_{k\in\mathcal{K}} p_{imk} x_{mk}$$

where p_{imk} is a penalty for not assigning shift k on day m to employee i, $\mathcal{P}_i$ is a set of patterns for employee i, $\#p$ is number of matches of pattern p in schedule $\mathbf{x}$ and c_p^{lb} (c_p^{ub}) is a penalty for violation $\#p$ of bound lb_p (ub_p respectively).

For solving this problem we proposed 3 different algorithms

- MIP model
- A^* based algorithm
- Branch-and-Bound algorithm with dominance rules

Upper Bound Prediction

Since pricing problem is solved frequently, we want to reuse information gathered in previous steps. We can predict upper bound by training following mathematical program

$$\min_{\mathbf{w},\mathbf{r}} \sum_{i\in\mathcal{D}} c_i^+ r_i^+ + c_i^- \max\{r_i^- - \epsilon, 0\} \quad (4)$$

subject to

$$\forall i \in \mathcal{D} : \quad \mathbf{w}^T \mathbf{x}_i + r_i^+ - r_i^- = y_i \quad (5)$$

$$\forall i \in \mathcal{D} : \quad r_i^+, r_i^- \geq 0 \quad (6)$$

$$\mathbf{w} \in \mathbb{R}^n \quad (7)$$

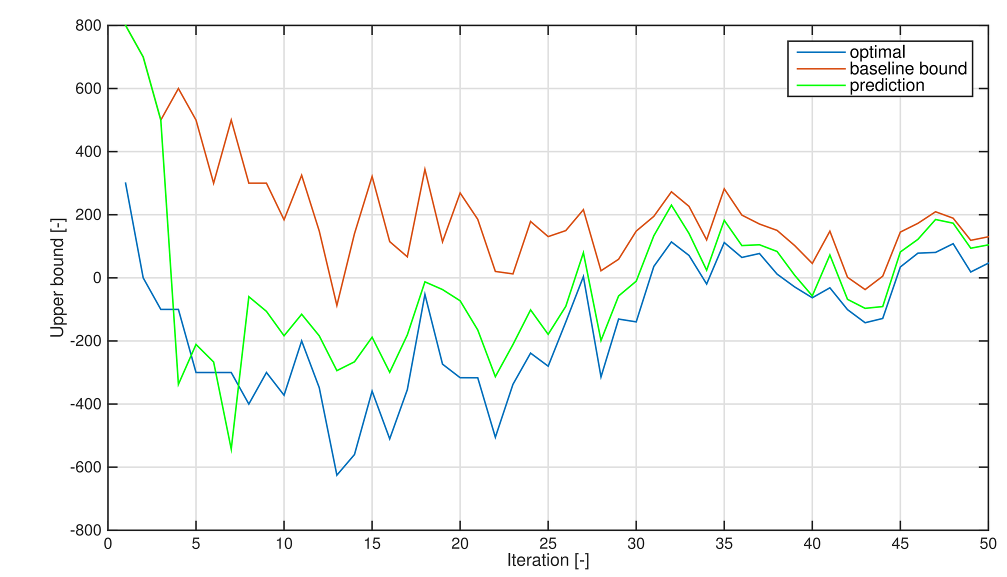

Figure 3: Example of tight bound prediction.

Results

Instance	Our runtime [s]	[Bur14] runtime [s]
Ozkarahan	**0.06**	5.1
Musa	**0.5**	5.1
LLR	**4.3**	5.8
Millar-DATA1	**0.29**	5.1
Millar-DATA1.1	**2.02**	5.1
WHPP	**20**	22.6
Azaiez	**3.3**	5.3
SINTEF	61	**15.5**
Motol-1	**991**	–

We compared our algorithm on challenging benchmark instances with recent publication [Bur14] dealing with NRP.

Contribution

- a new LP formulation of a master model allowing minimal and maximal staffing levels across subsets of skills and featuring a novel symmetry breaking technique
- a MIP based pricing solver with a number of improvements containing branching priorities and dynamical control of precision level based on the convergence of master model
- the proof that the decision version of the pricing problem with soft constraints is $\mathcal{NP}$-complete and is not polynomially approximable within ϵ
- new initiation heuristics and new primal heuristics for obtaining upper bounds
- a novel machine learning algorithm for upper bound prediction in the pricing problem