Comenius University, Bratislava

Faculty of Mathematics, Physics and Informatics

Equimatchable graphs on surfaces

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Abstract

A graph G is *equimatchable* if any matching of G is a subset of a maximum-size matching. From a general description of equimatchable graphs in terms of Gallai-Edmonds decomposition [Lesk, Plummer, and Pulleyblank, "Equimatchable graphs", *Graphs Theory and Combinatorics*, Academic press, London, (1984) 239-254.] it follows that any 2-connected equimatchable graph is either bipartite or factor-critical. In both cases, the Gallai-Edmonds decomposition gives no additional information about the structure of such graphs. It is well known that for any vertex v of a factor-critical equimatchable graph G and a minimal matching M_v that isolates v the components of the graph $G \setminus (M_v \cup \{v\})$ are all either complete or regular complete bipartite. We prove that for any 2-connected factor-critical equimatchable graph G, the graph $G \setminus (M_v \cup \{v\})$ has at most one component, and use this result to establish that the maximum size of such graphs embeddable in the orientable surface of genus g is $\Theta(g)$, improving on previous bound $O(g^{3/2})$. In addition, we bound the maximum size of k-degenerate 2-connected factor-critical graph. Moreover, for any non-negative integers g and k we provide a construction of arbitrarily large 2-connected equimatchable bipartite graphs with orientable genus g and a genus embedding with face-width k.

The structure of factor-critical equimatchable graphs with a cut-vertex or a 2-cut was determined in [J. Graph Theory, 10(4):439–448, 1986.]. We extend these results and for all $k \geq 3$ we describe the structure of factor-critical equimatchable graphs with a k-vertex-cut. More precisely, for every $k \geq 3$, we prove that if a k-connected equimatchable factor-critical graph G has at least $2k+3$ vertices and a k-cut S such that $G-S$ has two components with sizes at least 3, then $G-S$ has exactly two components and both are complete graphs. Consequently, if $k \geq 4$ such graphs have independence number 2. Additionally, we provide also a characterisation of k-connected equimatchable factor-critical graphs with a k-cut S such that $G-S$ has a component with size at least k and a component with size 1 or 2. Finally, we show that if every minimum cut S of an equimatchable factor-critical graph separates a component with size at most two, then the independence number can be arbitrarily high and the number of components can be as high as $|V(S)|$.

KEYWORDS: graph, matching, equimatchable, factor-critical, isolating matching, matching extendability, genus, embedding, representativity, sparse.

Abstrakt

Graf G sa nazýva *equimatchable*, ak sa každé jeho párenie dá rozšíriť na najväčšie párenie v G; teda každé párenie je podmnožinou nejakého najväčšieho párenia. Z charakterizácie equimatchable grafov pomocou Gallai-Edmonsovej dekompozície vypýva, že dvojsúvislé equimatchable grafy sú buď bipartitné alebo faktorovo-kritické. V oboch prípadoch Gallai-Edmondsová dekompozícia neposkytuje žiadne dodatočné informácie o štruktúre týchto grafov. Je známe že pre ľubovoľný vrchol v faktorovo-kritického equimatchable grafu G a každé minimálne izolujúce párenie M_v izolujúce vrchol v sú všetky komponenty grafu $G \setminus (M_v \cup \{v\})$ buď kompletné alebo regulárne kompletné bipartitné. V našej práci dokazujeme, že pre ľubovoľný dvojsúvislý faktorovo-kritický equimatchable graf G má graf $G \setminus (M_v \cup \{v\})$ najviac jeden komponent. Tento výsledok používame na určenie maximálneho počtu vrcholov takýchto grafov vnoriteľných do plochy rodu g na $\Theta(\sqrt{g})$, čím zlepšujeme doteraz známe ohraničenie $O(g^{3/2})$. Taktiež v práci ohraničujeme počet vrcholov k-degenerovaných dvojsúvislých faktorovo-kritických grafov. Pre ľubovoľné nezáporné prirodzené čísla g a k konštruujeme ľubovoľne veľké 2-súvislé bipartitné equimatchable grafy s orientovateľným rodom g a reprezentativitou k.

Štruktúra faktorovo-kritických equimatchable grafov vzhľadom na artikuláciu alebo dvojrez bola určená v článku [J. Graph Theory, 10(4):439–448, 1986.]. V tejto práci rozširujeme výsledky tohto článku a popisujeme štruktúru faktorovo-kritických equimatchable grafov s k-rezom pre všetky $k \geq 3$. Pre všetky $k \geq 3$ dokazujeme, že ak k-súvislý equimatchable graf má aspoň $2k+3$ vrcholov a k-rez S taký, že $G-S$ má dva komponenty s veľkosťou aspoň 3, potom $G-S$ má presne dva komponenty a oba sú kompletné grafy. Následne ukazujeme, že ak $k \geq 4$, potom takéto grafy majú číslo nezávislosti 2. Taktiež poskytujeme charakterizáciu k-súvislých faktorovo-kritických equimatchable grafov s takým k-rezom S, že $G-S$ má jeden komponent s veľkosťou aspoň k a druhý s veľkosťou 1 alebo 2. Nakoniec ukazujeme, že ak každý minimálný rez oddeľuje komponent s veľkosťou nanajvýš dva, potom graf môže obsahovať ľubovoľne veľkú nezávislú množinu vrcholov a počet komponentov $G-S$ môže dosiahnuť veľkosť $|V(S)|$.

KĽÚČOVÉ SLOVÁ: graf, párenie, equimatchable, faktorovo-kritický, izolujúce párenie, rozšíriteľnosť párení, vnorenie grafu, rod plochy, reprezentativita.

Contents

Introduction

In this thesis we investigate the structure of equimatchable graph. Equimatchable graphs are exactly the graphs in which one can always find maximum matching in linear time using greedy algorithm. Formally, a graph is called *equimatchable* if every its matching is a subset of a maximum matching. Equimatchable graphs constitute a classical topic of matching theory investigated for several decades since appearing in [12], [20], and [24]. In particular, Grünbaum [12] asked for a characterisation of all equimatchable graphs. The first step in this direction was a characterisation of all randomly-matchable graphs – equimatchable graphs with a perfect matching. Equimatchable graphs with a perfect matching were characterized by Sumner in [39] and all such graphs are isomorphic to K_{2n} or $K_{n,n}$. The fundamental work [19] provides a structural characterisation of equimatchable graphs without a perfect matching using Gallai-Edmonds decomposition. A particular consequences of this description are that there is a polynomial-time algorithm recognizing equimatchable graphs, and that every 2-connected equimatchable graph is either bipartite, or factor-critical. Despite a considerable effort was invested into the study of the relationship between equimatchability and other graph properties, the structure of equimatchable graphs is still not very well understood. A particular exception are equimatchable factor-critical graphs with 1 or 2-cuts, which were characterized in [8], and planar and cubic equimatchable graphs, which were characterized in [16].

In the paper [16] it is proved that if G is a 3-connected equimatchable planar graph, v a vertex of G, and M a *minimal* matching isolating v, then $G \setminus (V(M) \cup \{v\})$ is randomly matchable and connected, where a matching M is isolating a vertex v if $\{v\}$ is a component of $G \setminus V(M)$. Consequently authors showed that there are precisely twenty-three 3-connected equimatchable planar graphs. Later, Kawarabayashi and Plummer in paper [15] showed that for any fixed g, there are only finitely many 3-connected equimatchable graphs G embeddable in the surface of genus g with the property that either G is non-bipartite or the embedding has representativity at least three. The proof is based on a result that the maximum size of such a graph is at most $c \cdot g^{3/2}$, where c is a constant.

In Chapter 2 we improve the result of [15] and prove that if G is a 2-connected equimatchable factor-critical planar graph, v a vertex of G, and M a *minimal* matching isolating v, then $G \setminus (V(M) \cup \{v\})$ is randomly matchable and connected.

We use this result to show that if $f(g)$ is function that gives the maximum number of vertices of a 2-connected factor-critical equimatchable graph embeddable in the surface of orientable genus g, then $f(g) = \Theta(\sqrt{g})$. Moreover, we are able to show that every graph class with bounded degeneracy contains only a finite number of 2-connected factor-critical equimatchable graphs. We should note that classes with bounded expansion, excluded minor/topological minor, bounded degree, bounded tree-width, and bounded genus have all bounded degeneracy.

We also provide a construction of arbitrarily large 2-connected equimatchable bipartite graphs with genus g for any nonnegative integer g. We should note that every 2-connected equimatchable graph that is not bipartite is factor-critical.

The aim of Chapter 3 is to describe the structure of equimatchable factor-critical graphs with respect to their minimum cuts, thus extending the results of Favaron [8] to graphs with higher connectivity. Our main results can be described as follows. Let G be a k-connected equimatchable factor-critical graph with a k-cut S. If $G - S$ has a component with at least k vertices, then $G - S$ has exactly two components which are very close to complete or complete bipartite graphs. Furthermore, if $G - S$ has at least $k + 3$ vertices and two components with at least 3 vertices and $k \geq 4$, then the graph has independence number 2. This implies that for $k \geq 4$ a k-connected graph with odd number of vertices and a k-cut S such that $G - S$ has two components with sizes at least 3 is equimatchable and factor-critical if and only if it has independence number 2.

1

Definitions and Preliminaries

This chapter is devoted to a presentation of the basic concepts used in this thesis. We start with a summary of used graph-theoretic notation. In the second part of this chapter we define matching, equimatchable graph, and related concepts, present Edmonds-Gallai decomposition theorem and a characterization of equimatchable graphs based on this decomposition. The rest of this chapter consists from a short foundation of topology and topological graph theory.

1.1 Graphs

First we present some basic notation and definitions used throughout the text, for the concepts not defined the reader is referred to [4].

A *graph* is a pair $G = (V, E)$ of sets such such that $E \subseteq [V]^2$; thus the elements of E are 2-elements subsets of V. The elements of V are the *vertices* (or *node*, or *points*) of the graph G, the elements of E are its *edges* (or *lines*). The usual way to picture graph is by drawing a dot for each vertex and joining two vertices by a line if the corresponding two vertices form an edge. The vertex and edge set of a graph G are also denoted by $V(G)$ and $E(G)$, respectively. Graphs in topological graph theory are usually with loops and multiple edges. Since, in our thesis we work exclusively with matchings, we could excludes loops and multiple edges in graphs (see Note 1.2 on page 5). For a graph G and its vertex v, the set difference $G \setminus \{v\}$ is for brevity denoted by $G - v$. Moreover, for set difference between sets A and B we use $A \setminus B$ and $A - B$

interchangeably.

The number of vertices of a graph G is its *order*, written as $|G|$. The number of edges of graph G is denoted by $||G||$. A graph or a component is *even* if it has even number of vertices, otherwise it is *odd*. A vertex v is *incident* with an edge e if $v \in e$; then e is an edge at v. The two vertices incident with an edge are its *endvertices* or *ends*. An edge $\{x, y\}$ is usually written as xy (or yx). Two vertices x, y are adjacent, or neighbours, if xy is an edge of G. Two edges $e \neq f$ are adjacent if they have a common vertex as their end. If all vertices of graph are pair-wisely adjacent, then G is *complete*. A complete graph on n vertices is K_n. The *degree* $deg(v)$ of a vertex v is the number of edges incident with vertex v. By our definition the degree of a vertex v is equal to the number of neighbours of v. Set of neighbours of v is called *neighbourhood* (of v) and is denoted by $N(v)$. Let U be a set of vertices such that $U \subseteq V$. Then $N(U)$ denotes union of neighbourhoods of all vertices of U. If every vertex of the graph G has the same degree k, then G is said to be *k-regular*. A set of vertices or edges is said to be *independent* if no two of its elements are adjacent. The *independence number* of a graph G is the size of a largest independent set of G.

A graph is said to be *connected* if for any vertices a, b of G there is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices of graph such that $a = v_0$, $b = v_n$, and for each i the vertices v_i, v_{i+1} are adjacent. The maximal connected subgraphs of a graph G are called (connected) *components* of G. A graph is *k-edge-connected* for $k \geq 2$ if G is connected and for any set S of $k-1$ edges of G, the graph $G \setminus S$ is connected. Similarly, G is *k-vertex-connected*, or just *k-connected*, if it is connected and for every set S of $k-1$ vertices of G, the graph $G \setminus S$ is connected and it is not an isolated vertex. An edge e is a *bridge* if G is connected but $G \setminus e$ is not. Similarly, a vertex v is *cut-vertex* (or articulation) of G if G is connected but $G \setminus v$ is not.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is said to be a *subgraph* of graph G. If G' is a subgraph of a graph G such that $V' = V$ then G' is *spanning* subgraph of G. Subgraph $H = (V', E')$ of a graph $G = (V, E)$ is said to be *induced by* V' if for every edge $e \in E$ holds: if both ends of e are in V', then $e \in E'$. We denote the subgraph of graph G induced by vertex set U as $G[U]$, or just U, when it is clear that we mean a subgraph, not a vertex set. Unless evident from the immediate context otherwise, subgraphs in this thesis are always considered to be induced subgraphs.

Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is said to be *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes: vertices in the same partition class must be independent. Instead of '2-partite' one usually says *bipartite*. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete* (multipartite). Complete r-partite graph with partitions of sizes $n_1, \dots, n_r$ is denoted by $K_{n_1,\dots,n_r}$. Bipartite graphs are characterized by the following well-known property:

Proposition 1.1.1. *A graph is bipartite if and only if it contains no odd cycle.*

1.2 Matchings

A set M of independent edges in a graph $G = (V, E)$ is called a *matching*. Matching M is a matching of $U \subseteq V$ if every vertex of U is incident with an edge in M. The vertices in U are then called *matched* or *covered* (by M). Vertices not incident with an edge of M are *unmatched* or *uncovered*.

If A and B are subgraphs or sets of vertices of a graph G, then an edge, a set of edges, or a matching are said to be *between* A and B if every such edge has one endpoint in A and the other endpoint in B.

Note. In multigraphs, since a loop is considered to be adjacent to itself, they are banned to be in any matching. Only one edge between vertices u, v of graph G can be in matching. Therefore, for matchings it is important only if u and v are adjacent, and not how many edges are between them. Let a graph G be formed from a multigraph H by removing loops and replacing multi-edges by single edge. Then G has a matching M if and only if there exists a matching M' of H such that edge $xy \in M$ if and only if there is edge between vertices x and y in M'.

For a matching M, $|M|$ denotes the number of edges of M. A matching M in a graph $G = (V, E)$ is said to be *maximal* if any set $M' \subseteq E$, with $M' \supset M$ is not a matching in G. A matching M in G is *maximum* if, among all matchings in G, it is one with largest cardinality.

For a vertex v, a matching M is called a *matching isolating* v if $\{v\}$ is a component of $G - V(M)$. A matching M isolating a vertex v is called *minimal* if no subset of M isolates v.

A k-regular spanning subgraph is called *k-factor*. Thus, a subgraph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of $V(G)$. A non-empty graph $G = (V, E)$ is said to be *factor-critical* if G ha no 1-factor but for every vertex $v \in V$ the graph $G \backslash \{v\}$ has an 1-factor. A matching M that is an 1-factor is called *perfect* matching. If matching M leaves uncovered just one vertex, then M is said to be *near-perfect* matching.

The following theorem shows a necessary condition for bipartite graphs to have matching that saturates one partition.

Theorem 1.2.1 ([13])**.** *Let G be bipartite graph with partitions A and B. Then G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

1.2.1 Equimatchable graphs

A graph in which every matching extends to (i.e., is a subset of) a perfect matching is said to be *randomly matchable.* More generally, a graph in which every matching extends to (i.e., is a subset of) a maximum matching is called *equimatchable.*

Randomly matchable graphs were already characterized by Sumner in [39].

Theorem 1.2.2 ([39])**.** *A connected graph is randomly matchable if and only if $G = K_{n,n}$ or $G = K_{2n}$.*

Now we are ready to present Gallai-Edmonds (D, A, C) decomposition, which is very useful in the study of matchings in graphs, in particular in the study of equimatchable graphs.

For a graph $G = (V, E)$ denote by D the set of all vertices of G which are not saturated by at least one maximum matching of G. Let A be the neighbour set of D, i.e., the set of vertices in $V - D$ adjacent to at least one vertex in D. Finally $C = (V - D) - A$. Then (D, A, C) is called *Gallai-Edmonds decomposition* of the graph G. Using Gallai-Edmonds decomposition the following theorem describes the structure of all maximum matchings in graph G. The theorem was proved independently by Gallai ([9], [10]) and Edmonds ([6]).

Theorem 1.2.3 (Gallai-Edmonds Structure Theorem [9, 10, 6])**.** *Let G be a graph and (D, A, C) its Gallai-Edmonds decomposition. Then all the following conditions hold:*

(i) the components of the subgraph induced by D are factor-critical;

(ii) the subgraph induced by C has an 1-factor;

(iii) if M is a maximum matching of G, it contains a near-perfect matching of each component of D, a 1-factor of each component of C, and matches all vertices of A with vertices in distinct components of D;

(iv) the bipartite graph obtained from G by deleting the vertices of C and edges spanned by A and by contracting each component of D to a single vertex has a matching that saturates A.

(v) The size of any maximum matching is $\frac{1}{2}(|V| - \omega(D) + |A|)$, where $\omega(D)$ is the number of componenents of $G[D]$.

Using the previous theorem it is easy to prove the next lemma stated as Lemma 1 in [19].

Lemma 1.2.4. *Let G be a connected equimatchable graph with no perfect matching, having Gallai-Edmonds decomposition (D, A, C). Then $C = \emptyset$ and A is an independent set in G.*

The following characterization of equimatchable graphs was proved in [19].

Theorem 1.2.5 ([19])**.** *Let G be a connected equimatchable graph without a perfect matching. Let (D, A, C) be its Gallai-Edmonds decomposition and suppose $A \neq \emptyset$. Let D_i denote any component of D with $|D_i| \geq 3$. Then all of the following conditions hold:*

(1) Component D_i must be one of following types of graphs:

I. $D_i \cong K_{2m+1}$ for some $m \geq 2$ and every point of D_i is joined to exactly one common point $a \in A$.

II. D_i contains a cut-vertex d_i of G (called hook *of D_i) which is the only vertex of D_i adjacent to a point of A. Let $H_i^1, \ldots, H_i^r$ be the components of $D_i - d_i$. Consider* *any one of these, say H_i^j. There are two possibilities:* (a) $H_i^j \cong K_{2m}$ *for some* $m \geq 1$ *and at least two edges join* d_i *to* H_i^j*, or* (b) $H_i^j \cong K_{m,m}$ *for some* $m \geq 1$ *and if* (U, W) *is the bipartition of* H_i^j*, at least one edge joins* d_i *to a vertex* u *of* U *and at least one edge joins* d_i *to a vertex* w *of* W.

III. *At least two vertices of* D_i *are adjacent to points of* A *and at least one vertex of* D_i *is adjacent to no point of* A*. In this case there is a vertex* $a \in A$ *such that* a *separates* D_i *from rest of graph. Here we have four subcases. If* D_i *contains exactly two vertices* y_1 *and* y_2 *of attachment to* a*, then* D_i *must be one of following three types:* (a) D_i *is* K_3*;* (b) $(D_i - y_1 - y_2)$ *is a complete bipartite graph* $Kr, r-1$ *where* $r \geq 2$*, and if* (U, W) *is the bipartition of* $D_i - y_1 - y_2$ *where* $|U| = r$*, then* y_1 *and* y_2 *are both adjacent to all points of* U *and to each other;* (c) $(D_i - y_1 - y_2)$ *is* K_{2r-1}*,* $r \geq 2$*,* y_1 *and* y_2 *are both adjacent to all vertices of* $D_i - y_1 - y_2$ *and* y_1 *and* y_2 *may or may not be adjacent to each other. The fourth subcase may be stated as follow:* (d) *if* D_i *has between* 3 *and* $|D_i| - 1$ *points of attachment to* a*, then* D_i *is* K_{2r-1} *for some* $r \geq 3$.

(2) Suppose we delete all type II and type III components of D from G and contract all type I components to single points. Then there is a matching of resulting (bipartite) graph G' *which covers all vertices of* A *and* G' *is equimatchable.*

The next theorem, converse of Theorem 1.2.5 was proved in [19].

Theorem 1.2.6. *Let* G *be connected graph without a perfect matching, which is not factor-critical and which has Gallai-Edmonds decomposition* (D, A, C)*. Suppose*
(1) $C = \emptyset$*; and*
(2) A *is independent set.; and*
(3) All components of D *are singletons or of types I, II, or III as described in Theorem 1.2.5.; and*
Let G_1 *be the bipartite graph obtained from* G *by shrinking (contracting) all components of* D *to singletons and let* G_1' *be the graph obtained from* G_1 *by deleting all points corresponding to type II and III components of* D*. Suppose:*
(4) G_1' *is equimatchable graph and* $|A| \leq \frac{1}{2}|V(G_1')|$.

Then G *is equimatchable.*

Theorem 1.2.7. *A connected bipartite graph $G = (U, W)$ with $|U| \leq |W|$, is equimatchable if and only if, for all $u \in U$, there exists a non-empty $X \in N(u)$ such that $|N(X)| \leq |X|$.*

1.3 Surfaces and Embeddings

In this part we briefly introduce basic concepts of topological graph theory - topological surfaces and embeddings of graphs in surfaces. Most of definitions and theorems in this section is from [11] and from [40]. For a deeper account of topology, the reader is referred to [1].

An embedding of a graph in a surface generalizes the concept of an embedding of a graph in the plane. From a visual point of view, we can imagine embedding as a drawing of the graph on a sphere, torus, double-torus or a similar surface.

Formally, any graph can be presented by a topological space in following sense. Each vertex is represented by a distinct point and each edge by a distinct arc, homeomorphic to a closed interval $[0, 1]$. Naturally, the boundary points of an arc represent the ends of the corresponding edge. (Of course, interiors of arcs are mutually disjoint and do not meet the points representing vertices.) Such a space is called *topological representation* of the graph G.

Graphs G and H are said to be *homeomorphic* if they have respective subdivisions G' and H' such that G' and H' are isomorphic.

The central concern of topological graph theory is the placement of graphs on surfaces. A topological space M is called *n-manifold* if M is Hausdorff(see [1]) and can be covered by countably many open sets, each of which is homeomorphic either to the n-dimensional open ball

$$\{(x_1, \dots, x_n)|x_1^2 + \cdots + x_n^2 < 1\}$$

or the n-dimensional half-ball

$$\{(x_1, \dots, x_n)|x_1^2 + \cdots + x_n^2 < 1, x_n \geq 0\}.$$

A manifold is *closed* if it is compact and its boundary is empty. By surface we usually mean closed, connected 2-manifold, such as the sphere, the torus, or the Klein bottle.

We define an embedding of a graph in a surface. Let G be a graph and S a surface. An *embedding* is a continuous one-to-one function $\Pi : G \to S$. Usually, we consider our graphs to be subsets of the surface S, and the function $\Pi : G \to S$ is inclusion map. The embedding is then denoted simply $G \to S$.

Given an embedding $G \to S$, the components of $S - G$ are called *regions*. Regions are also called *faces* of embedding. If each region is homeomorphic to an open disc, the embedding is said to be 2-*cell (or cellular) embedding*. The closure in surface S of a region in the 2-cell embedding $G \to S$ need *not* be homeomorphic to closed disc. If there exists a boundary walk containing vertices x and y, then we say that vertices x and y are on same face of embedding $G \to S$.

Each face of an embedding $G \to S$ has two possible directions for its boundary walk. A face is assigned an *orientation* by choosing one of these two directions. An *orientation of embedding* $G \to S$ is an assignment of orientations to all faces so that adjacent regions induce opposite direction on every common edge. If a graph G is 1-skeleton of a triangulation of surface S, then orientation of embedding $G \to S$ is called *orientation of triangulation*. A surface is *orientable* if for every graph G there exists an embedding $G \to S$ with an orientation. If every embedding of a graph to a surface does not have an orientation, then the surface is said to be *non-orientable*. In this work, we will deal exclusively with orientable surfaces.

Given an orientable surface, we can add *handle* to it in such a way that the resulting object is an orientable surface. For example, we can obtain the torus by adding a handle to the sphere. In general, starting with the sphere S_0 we can add g handles to it. The resulting surface is called a sphere with g handles and it is denoted S_g . The number g is then called the *orientable genus* of the surface. The following crucial theorem asserts that these are essentially the only orientable surfaces.

Theorem 1.3.1. *The surfaces S_g, $g = 0, 1, 2, \dots$ are pairwise non-homeomorphic and every closed orientable surface is homeomorphic to one of them.*

The minimum g such that there exists embedding $G \to S_g$ is called *genus of graph* and is denoted $\gamma(G)$. The maximum such g that there exists cellular embedding $G \to S_g$ is denoted $\gamma_M(G)$.

Theorem 1.3.2 ([5]). *A connected graph G has a 2-cell embedding in S_g if and only if $\gamma(G) \leq g \leq \gamma_M(G)$.*

In our thesis we will use the following theorem about genus of complete and complete bipartite graphs frequently. The theorem can be found in chapter 6 of [40].

Theorem 1.3.3 ([32, 29, 30, 31]). *The orientable and nonorientable genera of complete and complete bipartite graphs are given by the following formulae:*

$$\begin{aligned}
\gamma(K_n) &= \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, n \geq 3; \\
\tilde{\gamma}(K_n) &= \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, n \geq 3 \quad \text{and } n \neq 7,\ \tilde{\gamma}(K_7) = 3; \\
\gamma(K_{m,n}) &= \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, m, n \geq 2; \\
\tilde{\gamma}(K_{n,m}) &= \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil, m, n \geq 2.
\end{aligned}$$

In addition, we mention the next theorems about the maximum genus of complete and complete bipartite graphs.

Theorem 1.3.4 ([27]). *Let $G = K_{m,n}$. Then*

$$\gamma_M(G) = \left\lfloor \frac{(m-1)(n-1)}{2} \right\rfloor.$$

Theorem 1.3.5 ([28]). *Let $G = K_n$. Then*

$$\gamma_M(G) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor.$$

Let $\Pi : G \to S$ be an embedding. Denote number of vertices of G p, number of edges q and number of faces in embedding Π as r. From this time forth in this section we will be using former denotation for number of vertices, edges and faces in Π.

Let Π be an embedding of a connected graph into a closed, connected surface. The *Euler characteristic of* Π is the value $p - q + r$, and it is denoted $\chi(\Pi)$. The following famous formula shows that for every standard surface the value of Euler characteristic is independent from the choice of graph and of a cellular embedding.

Theorem 1.3.6 (The Euler-Pointcaré formula)**.** *Let* $G \to S$ *be a* 2*-cell embedding, for any* $g = 0, 1, 2, \dots$*. Then* $\chi(G \to S) = 2 - 2g$*.*

The Euler-Pointcaré formula is often used in conjunction with relationship between the numbers of edges and faces to prove that certain graphs cannot be embedded into the surface S_g.

We now present a well-known upper bound on the number of faces of an embedding of a simple graph.

Theorem 1.3.7. *Let* G *be a simple graph with* q *edges embedded with* r *faces. Then* $2q \geq 3r$*.*

Proof. As G is simple, any face of the embedding has length at least 3. The result follows from the fact that the union of face boundaries contains every edge precisely twice. □

Actually, if we have a graph with given girth then following theorem holds:

Theorem 1.3.8. *Let* G *be connected graph that is not a tree and let* $\Pi : G \to S$ *be an embedding. Then* $2q \geq girth(G) \cdot r$*.*

Face-width, sometimes called also *representativity* or *planar-width*, of an embedding Π in a surface S is the minimum number of faces of Π whose union contains a noncontractible cycle in the surface S. Several equivalent definitions and further details about face-width can be found in [25].

1.3.1 Rotation systems

Define a 1*-band* to be a topological space b together with homeomorphism $h : I \times I \to b$, where I denotes the unit interval $[0, 1]$. The arcs $h(I \times \{j\})$ for $j = 0, 1$ are called *ends* of band b and the arcs $h(\{j\} \times I)$ for $j = 0, 1$ are called *sides* of band b. A 0*-band* and a 2*-band* are simply homeomorphs of the unit disc. A *band decomposition* of the surface S is collection B of 0-bands, 1-bands, 2-bands satisfying these conditions:

(1) Different bands intersect only along arcs in their boundaries.
(2) The union of all the bands is S

(3) Each end of each 1-band is contained in a 0-band.
(4) Each side of each 1-band is contained in a 2-band.
(5) The 0-bands are pairwise disjoint and the 2-bands are pairwise disjoint.

Corresponding *reduced band decomposition* B omits the 2-bands. Note that, in embedding $G \to S$ 0-bands represents vertices of G, 1-bands represents its edges and 2-bands represents regions of embedding. To describe a embedding $G \to S$ or equivalently its band decomposition are 2-bands not really needed to define, since the union of 1.bands and 0-bands is surface with boundary, and since is essentially only one way how to fill in the faces to complete to closed surface.

A band decomposition is called *locally oriented* if each 0-band is assigned an orientation. Then 1-band is called *orientation-preserving* if direction induced on its ends by adjoining 0-bands are the same as those induced by one of two possible orientation of 1-band; otherwise 1-band is called *orientation-reversing*. An edge e in graph embedding associated with locally oriented band decomposition is said to have *(orientation) type 0* if its corresponding 1-band is orientation-preserving and *(orientation) type 1* otherwise.

To describe a graph embedding $G \to S$ or equivalently its band decomposition, we need to specify only how the ends of 1-bands are attached to the 0-bands. We define *rotation* at a vertex v of graph to be ordered list, unique up to cyclic permutation, of the edges incident on that vertex. Let a *rotation system* on a graph be an assignment of a rotation to each vertex and a designation of orientation type for each edge. Then the preceding discussion can be summarized by following theorem.

Theorem 1.3.9. *Every rotation system on a graph G defines (up to equivalence of embeddings) a unique locally oriented graph embedding $G \to S$. Conversely, every locally oriented graph embedding $G \to S$ defines a rotation system for G.*

From now on we will use the terms embedding and rotation system interchangeably.

Given a rotation system for a graph, one frequently needs to obtain a listing or enumeration of boundary walks of the reduced faces. We first introduce some helpful terminology. If rotation at vertex v is $\ldots de \ldots$, then we say d is *the edge before e at v*, that e is *the edge after d at v*, and that edge pair (d, e) is *corner at v with second edge e*.

To enumerate boundary walks of reduced faces we use following algorithm *Face Tracing Algorithm*.

Face Tracing Algorithm

Assume that given graph G has not any vertex of degree two.

(1) Choose an initial vertex v_0 of G and a first edge e_1 incident on v_0. Let v_1 be the other endpoint of e_1.

(2) If the walk traced so far ends with edge e_i at vertex v_i then the next edge e_{i+1} in the boundary walk is the edge after (resp., before) e_i at v_i if e_i is type 0 (resp., type 1).

If the next two edges in the walk would not be e_1 and e_2 then

(3) Go to step (2).

Else

(4) The boundary walk is finished at edge e_n.

(5) If there is a corner at any vertex v that does not appear in any previously traced faces, then choose as initial vertex v and as the first edge second edge of this corner at v, and go to step (2)

(6) If there are not unused corners, then all faces have been traced.

Suppose graph G has some vertices with degree 2. Then we just find the graph H, without valent 2 vertices, such that G is subdivision of H. Then we use face-tracing algorithm on H and subdivide edges to correspond with graph G.

2

2-connected equimatchable graphs on surfaces

In this chapter we investigate the structure of factor-critical equimatchable graphs with respect to a minimal matching isolating a vertex. A matching M is *isolating* a vertex v if $\{v\}$ is a component of $G \setminus V(M_v)$. We use structural results to bound the maximum number of 2-connected equimatchable graphs embeddable into fixed surface and 2-connected equimatchable graphs with bounded degeneracy.

The relationship between embeddings of graphs and matching extensions was extensively studied, see for instance [3], [14], or [21]. The characterisation of equimatchable graphs in [19] implies that any 2-connected equimatchable graph is either bipartite or factor-critical. A bipartite graph cannot be factor-critical, since otherwise it would have an odd number of vertices and removing a vertex from the smaller partite set cannot yield a graph with a perfect matching. Therefore, these two classes are disjoint. All 3-connected planar equimatchable graphs are characterised in [16] – there are 23 such graphs and none of them is bipartite. Let G be a 3-connected equimatchable graph with an embedding Π in the surface of genus g. In [15] it is proved that if G is either factor-critical, or bipartite and Π has face-width at least 3, then the number of vertices of G is bounded from above by $c \cdot g^{3/2}$ for some constant c. The proof uses the fact that there is no such bipartite graph at all and proceeds to restrict the size of equimatchable factor-critical graphs embeddable in a fixed surface. First it is shown that if a 3-connected graph has many vertices (a number linear in the genus of the graph), then it has a vertex v isolated by a matching M_v of size at most 4. The proof is finished by showing that $G \setminus (V(M_v) \cup \{v\})$ has at most $\binom{8}{3}(4g+3)$ components.

To bound the maximum size of equimatchable factor-critical graphs embeddable in a fixed surface, we employ a slightly different strategy: while we allow larger isolating matchings, we use a more precise description of $G \setminus (V(M_v) \cup \{v\})$ given in our main result, Theorem 2.1.3, which implies that it has at most one component. As a complete or complete bipartite graph embeddable in the surface of genus g has at most $O(\sqrt{g})$ vertices, it suffices to bound the size of isolating matchings. Note that any vertex of degree d admits an isolating matching of size at most d. The last ingredient of our proof is Lemma 2.2.7 showing that either the total number of vertices of the graph, or the minimum degree, is sufficiently small.

Concerning the methods used in this chapter, while we repeatedly use the characterisation of randomly matchable graphs from [39], the Gallai-Edmonds decomposition is not used beyond the fact that every 2-connected equimatchable graph is either bipartite or factor-critical. The constants in the orientable and the nonorientable case are different, hence we state our results explicitly for both cases. However, most of the proofs are virtually identical and in such cases, we omit the proof of the nonorientable case.

The chapter is organized as follows. In Section 2.1 we present a proof of our main result stating that the graph $G\setminus(V(M_v)\cup\{v\})$ is connected for any 2-connected factor-critical equimatchable graph G and a minimal matching M_v isolating a vertex v. Section 2.2 is devoted to lower and upper bounds on the maximum size of an equimatchable graph embeddable in a fixed surface.

Most of the material in this chapter is based on [7].

2.1 Isolating matchings 2-connected equimatchable graphs

This section is devoted to the proof of our main result stated as Theorem 2.1.3. We start with two lemmas concerning isolating matchings.

Lemma 2.1.1. *Let G be a factor-critical graph. For every vertex v of G there is a matching $M_v \subseteq E(G)$ isolating v such that $|M_v| \leq \deg(v)$.*

Proof. Since G is factor critical, the graph $G' = G - v$ has a perfect matching M'.

Clearly, every neighbour of v is incident with exactly one edge of the matching M'. Consider a set $M \subseteq M'$ such that M contains precisely those edges from M' that are incident with at least one neighbour of v. Then M is the desired matching M_v containing at most $\deg(v)$ edges and isolating v. □

Favaron [8, Theorem 1.1] proved that any connected factor-critical equimatchable graph G with a cut-vertex contains precisely one cut-vertex v and every component of $G - v$ is either K_{2n} or $K_{n,n}$. For equimatchable factor-critical graphs with a 2-cut $\{u, v\}$, it is still possible to give a description of the structure of $G' = G \setminus \{u, v\}$, albeit it is more complicated: G' has exactly two components and these components are almost complete or complete bipartite, see [8, Theorem 2.2] for the precise statement and details. Removing isolating matchings instead of vertex-cuts allows us to obtain a similar description for graphs with arbitrary connectivity in the lemma below. The underlying idea of its proof is well known, in particular, it was applied in [16] and [15] to prove more specific variants of the result.

Lemma 2.1.2. *Let G be a connected factor-critical equimatchable graph and M a minimal matching isolating v. Then every component of $G \setminus V(M)$ except $\{v\}$ is isomorphic with either K_{2n} or $K_{n,n}$ for some integer n.*

Proof. Let $G' = G \setminus (V(M) \cup \{v\})$ and denote by M' any maximal matching of G'. Clearly, $M = M' \cup M_v$ is a maximal matching of G. The graph G is factor-critical and equimatchable, hence M leaves only the vertex v uncovered and M' must be a perfect matching of G'. Since arbitrary maximal matching M' of G' is a perfect matching of G', G' is randomly matchable and by [39] all of its components are either complete with even number of vertices or complete regular bipartite. □

Note that since G is factor-critical, there always exists a matching isolating any fixed vertex v of G.

We say that a subgraph H_1 (such as a vertex, edge, or component) of a graph G is *linked* with other subgraph H_2 of same graph G if there are vertices k_1 of H_1 and k_2 of H_2 such that $k_1k_2 \in E(G)$. We are now ready to prove our main result, which sharpens Lemma 2.1.2 by showing that G' has only one component and generalizes [16, Lemma 1.6], which proves that G' has only one component if G is 3-connected and planar.

Theorem 2.1.3. *Let G be a 2-connected, factor-critical equimatchable graph. Let v be a vertex of G and M_v a minimal matching isolating v. Then $G \setminus (V(M_v) \cup \{v\})$ is isomorphic with K_{2n} or $K_{n,n}$ for some nonnegative integer n.*

Proof. We prove the theorem by a series of claims. Let $G' = G \setminus (V(M_v) \cup \{v\})$.

Claim 1. If xy is an arbitrary edge of matching M_v, then x and y cannot be linked to different components of G'.

Proof of Claim 1. We prove the claim by contradiction. Let C and D be different components of G' and suppose that x is adjacent to a vertex x' of C and y is adjacent to a vertex y' of D. Let M be defined by $M = (M_v \setminus \{xy\}) \cup \{xx', yy'\}$. It is easy to see that M is a matching of G. Furthermore, $C - x'$ and $D - y'$ are components of $G \setminus M$. From Lemma 2.1.2 follows that C and D have even number of vertices and hence both $C - x'$ and $D - y'$ have odd number of vertices. It follows that any maximal matching M' such that $M \subseteq M'$ leaves uncovered at least one vertex of both $C - x'$ and $D - y'$. This is a contradiction with the fact that G is equimatchable and factor-critical.

Claim 2. Let C be a component of G' and xy an edge of matching M_v such that x is linked to some vertex x' from C. Then y is linked either to v or to some vertex y' of C such that $y' \neq x'$.

Proof of Claim 2. Suppose that y is linked neither with C nor with v. Let M be defined by $M = (M_v \setminus \{xy\}) \cup \{xx'\}$. It is easy to see that M is a matching of G. As all neighbours of v are covered by M, any maximal matching M' of G such that $M \subseteq M'$ leaves v uncovered. Since x is linked with C, by Claim 1 y cannot be linked to any other component of G'. According to our assumption, y is not linked with v or C. Therefore, M' leaves uncovered both v and y. This is a contradiction with the fact that G is equimatchable and factor-critical, which completes the proof of the claim.

Claim 3. For any edge e of M_v linked with a component C of G', there are two independent edges joining the endvertices of e with v and C, respectively.

Proof of Claim 3. Let $e = xy$ and suppose that x is linked with a vertex x' of C. By Claim 2, y is linked either with v or with some vertex y' of C. If y is linked with v, then xx' and yv are the two desired edges and we are done. If y is not linked with v, then by the minimality of M_v v is linked with x. In this case xv and yy' are the desired edges, which completes the proof.

Claim 4. Let C be an arbitrary component of G' and xy an edge of M_v linked with C. If G' has at least two components, then there are two independent edges joining x and y with C.

Proof of Claim 4. Without loss of generality assume that x is adjacent to a vertex x' of C and suppose to the contrary that y is not adjacent to a vertex of C different from x'. Let D be a component of G' different from C. Since G is 2-connected, D is linked with at least two vertices of $G \setminus V(D)$. Furthermore, the fact that v is not linked with D implies that these two vertices must be vertices of M_v. Because x is linked with C, from Claim 1 we get that y cannot be linked with D and thus at least one of the vertices of M_v linked with D is different from both x and y. Let x_1y_1 be an edge of M_v linked with D such that $x_1y_1 \neq xy$. According to Claim 3 we can assume that x_1 is adjacent to a vertex x_1' from D and y_1 is adjacent to v. It is clear that the set M defined by $M = (M_v \setminus \{xy, x_1y_1\}) \cup \{xx', x_1x_1', y_1v\}$ is a matching of G. Claim 1 implies that y is not linked with any component of G' different from C and in particular, it is not linked with D. According to our assumption, y is not adjacent to any vertex of $C - x'$. It follows that any maximal matching M' such that $M \subseteq M'$ leaves uncovered y and one vertex of both C and D. This contradicts equimatchability and factor-criticality of G and completes the proof of the claim.

Claim 5. Let e and f be two edges of M_v linked with two different components of G'. Then e and f are not linked.

Proof of Claim 5. Let $e = x_1y_1$ and $f = x_2y_2$. Assume that e is linked with a component C of G' and f is linked with a component D of G'. Claim 4 implies that both x_1 and y_1 are linked with C and both x_2 and y_2 are linked with D. Suppose to the contrary that that e and f are linked; we can assume that they are linked by edge x_1x_2. Let y_1' be a vertex of C adjacent to y_1 and y_2' a vertex of D adjacent to y_2. Clearly, the set M defined by $M = (M_v \setminus \{x_1y_1, x_2y_2\}) \cup \{x_1x_2, y_1y_1', y_2y_2'\}$ is a matching of G and any maximal matching M' such that $M \subseteq M'$ leaves unmatched v and at least one vertex of both C and D, again contradicting the equimatchability and factor-criticality of G.

Claim 6. Let e, f_1, and f_2 be edges of M_v and C and D two different components of G' such that C is linked with f_1 and D is linked with f_2. If e is not linked with C, then it is not linked with f_1.

Proof of Claim 6. Let $e = uw$, $f_1 = x_1y_1$, and $f_2 = x_2y_2$, and for the contrary suppose that e is linked with f_1. Since e and f_1 are linked, by Claim 5 e is not linked to any component of G' different from C. Moreover, by our assumption e is not linked with C. By Claim 4 there are two independent edges joining f_1 and C and two independent edges joining f_2 and D. Therefore, we can assume that u is linked with x_1. As M_v is minimal, f_2 is linked with v; let x_2 be adjacent to v. Let y_1' be a vertex of C adjacent to y_1 and y_2' a vertex of D adjacent to y_2. It is clear that the set M defined by $M = (M_v \setminus \{e, f_1, f_2\}) \cup \{ux_1, y_1y_1', vx_2, y_2y_2'\}$ is a matching of G. Since e is not linked with any component of G', any maximal matching M' of G such that $M \subseteq M'$ leaves unmatched the vertex w and one vertex of both C and D, which contradicts the fact that G is equimatchable and factor-critical.

Claim 7. If G' has at least two components, then v is a cutvertex.

Proof of Claim 7. Our aim is to show that in $G - v$ there is no path between arbitrary two components of G'. We proceed by contradiction: suppose there is such a path and among all such paths, choose a path that minimizes the number k of edges of M_v incident with it. Denote one of the paths with k minimal by P and by C and D the components of G' joined by P. From the fact that C and D are components of G' follows that they cannot be linked directly, and consequently $k > 0$. Let e and f be the first, respectively the last, edge of M_v incident with P. As no other component of G' is linked with either C or D, we get that e is linked with C and f is linked with D. From Claim 4 follows that both endvertices of e are linked with C and then Claim 1 implies that e is not linked with D. Therefore, e and f are distinct and $k > 1$. Notice that $k = 2$ is equivalent with e and f being linked, which is not possible due to Claim 5. Suppose that $k \geq 3$. By the minimality of k, there is an edge a of M_v such that a is linked with e, but not with C. However, this contradicts Claim 6 and hence $k \geq 3$ is not possible. We conclude that any path between C and D contains v. Since G is connected, there is at least one such path. Consequently, v is a cutvertex of G, which completes the proof of the claim.

From the fact that G is 2-connected and from Claim 7 it follows that G' has only one component. Lemma 2.1.2 implies that this component is either K_{2n} or $K_{n,n}$, which completes the proof. □

The characterisation of equimatchable factor-critical graphs with a cut-vertex in [8] implies that in such graphs $G \setminus V(M_v)$ can have arbitrarily-many components and therefore, Theorem 2.1.3 cannot be extended to graph that are not 2-connected.

Now, we are able to show that every graph class with bounded degeneracy contains only a finite number of 2-connected factor-critical equimatchable graphs. A graph G is said to be k-*degenerate* if every subgraph of G has a vertex of degree at most k. The *degeneracy* of graph is the smallest number k such that graph is k-degenerate and in a sense it measure a sparsity of the graph. If a graph G is k-degenerate then G contains at most $k \cdot |V(G)|$ edges.

Theorem 2.1.4. *Let G be a 2-connected factor-critical equimatchable graph with degeneracy at most k. Then $|V(G)| \leq 4k + 1$.*

Proof. Since G has degeneracy at most k, then by definition G has a vertex v with degree at most k. Let M_v be a minimal matching that isolates v. By Lemma 2.1.1 M_v covers at most $2k$ vertices. Let $G' = G \setminus (V(M_v) \cup \{v\})$. By Theorem 2.1.3 G' has at most one component, this component is randomly matchable. Since G' is a subgraph of the graph with a degeneracy at most k, it has a vertex with degree at most k. This together with the fact that G' is a complete or a regular complete bipartite graph imply that G' has at most $2k$ vertices. The proof is now complete. □

A graph H is called a topological minor of a graph G if a subdivision of H is isomorphic to a subgraph of G.

Corollary 2.1.5. *Let $\mathcal{H}$ be a non-empty set of graphs and $\mathcal{C}$ a family of graphs with $\mathcal{H}$ as a set of forbidden topological minors. Then there are only a finite number of 2-connected factor-critical equimatchable graph in $\mathcal{C}$.*

Proof. Mader [23] shows that for given $t \in \mathbb{N}$, there exists a constant c_t depending only on t such that every graph G on n vertices with at least $c_t n$ edges contains a subdivision of the complete graph K_t. Let H be an arbitrary graph from $\mathcal{H}$. Clearly, $K_{|V(H)|}$ contains H as a minor. Since, every subgraph K of a graph G from $\mathcal{C}$ does not have a H as a minor, it has at most $c_{|V(H)|}|V(K)|$ edges and hence a vertex with degree $2c_{|V(H)|}$. Therefore, every graph in $\mathcal{C}$ is $2c_{|V(H)|}$-degenerate and by Theorem 2.1.4 there are only finitely many 2-connected factor-critical equimatchable graphs with degeneracy bounded by a constant. This completes the proof. □

A graph H is called a minor of a graph G if a graph isomorphic to H can be obtained from G by deleting edges, vertices, and contracting edges. It is easy to see that if G

contains H as a topological, it contains H as a minor. Therefore, we get the following corollary.

Corollary 2.1.6. *Let $\mathcal{H}$ be a non-empty set of graphs and $\mathcal{C}$ a family of graphs with $\mathcal{H}$ as a set of forbidden minors. Then there are only a finite number of 2-connected factor-critical equimatchable graph in $\mathcal{C}$.*

By Kuratowski's theorem, planarity can be expressed by forbidding the minors K_5 and $K_{3,3}$. Later, Robertson and Seymour [33] showed that for every surface S there exists a finite set of graphs $H_1, \dots, H_n$ such that a graph is embeddable in S if and only if it contains none of $H_1, \dots, H_n$ as a minor. As a consequence of this and Corollary 2.1.6, we get the result of [15] for 2-connected factor-critical equimatchable graphs.

Corollary 2.1.7. *Let S be a surface of fixed genus. Then there are only finitely many 2-connected factor-critical equimatchable graph embeddable in S.*

2.2 Size of 2-connected equimatchable graphs on surfaces

The aim of this section is to obtain good lower and upper bounds on the maximum size of equimatchable factor-critical graphs embeddable in the surface of arbitrary fixed genus using Theorem 2.1.3. We start by showing that there are arbitrarily large equimatchable factor-critical graphs with a cutvertex and any given genus.

Proposition 2.2.1. *For any nonnegative integers g, h, and k there exist connected factor-critical equimatchable graphs G and $\tilde{G}$ with at least k vertices such that G has orientable genus g and $\tilde{G}$ has nonorientable genus h.*

Proof. Let n be an integer such that K_{2n+1} has orientable genus g and v an arbitrary vertex of K_{2n+1}. Take k copies of the triangle K_3 and designate one vertex in each copy. It is easy to verify that the graph obtained by vertex amalgamation of K_{2n+1} at v and k triangles at the designated vertices is a connected factor-critical equimatchable graph with genus g and at least k vertices. The proof of the nonorientable case is analogous. □

It is easy to see and well-known that any complete bipartite graph $K_{m,n}$ is equimatchable.

Proposition 2.2.2. *For any integers m and n such that $m \geq n$ the complete bipartite graph $K_{m,n}$ is equimachable and its maximum matching has size n.* □

The next three results yield a construction of large 2-connected equimatchable factor-critical graphs embeddable in any fixed surface.

Lemma 2.2.3. *Let u and v be adjacent vertices of $K_{n,n}$ and x and y different vertices from the larger partite set of $K_{m+1,m}$ for some m and n. Then the graph G defined by $G = K_{n,n} \cup K_{m+1,m} \cup \{ux, vy\}$ is factor-critical and equimatchable.*

Proof. Denote by H_1 the copy of $K_{n,n}$ and by H_2 the copy of $K_{m+1,m}$ in G, thus $G = H_1 \cup H_2 \cup \{ux, vy\}$. First, we show that G is factor-critical, that is, the graph $G - w$ has a perfect matching for any vertex w of G. Denote by A and B the larger, respectively the smaller partite set of H_2. We distinguish three cases.

Case 1: w is a vertex of H_1. We can assume that w is in same partite set as v. Clearly, there is a perfect matching M_1 of $H_1 \setminus \{u, w\}$ and a perfect matching M_2 of $H_2 - x$. It follows that matching M defined by $M = M_1 \cup M_2 \cup \{ux\}$ is a perfect matching of $G - w$.

Case 2: w is a vertex of A. Take any perfect matching M_1 of H_1 and any perfect matching M_2 of $H_2 - w$. The matching M defined by $M = M_1 \cup M_2$ is clearly a perfect matching of $G - w$.

Case 3: w is a vertex of B. Take any perfect matching M_1 of $H_1 \setminus \{u, v\}$ and any perfect matching M_2 of $H_2 \setminus \{w, x, y\}$. It is easy to see that the matching M defined by $M = M_1 \cup M_2 \cup \{ux, vy\}$ is a perfect matching of $G - w$.

Now we show that G is equimatchable by proving that any matching M of G is a subset of a maximum matching. As G is factor-critical, any maximum matching of the graph G leaves precisely one vertex uncovered. We distinguish three cases according to which of the edges ux and vy lie in M.

Case 1: neither ux nor vy is an edge of M. Clearly, M is a disjoint union of matchings M_1 of H_1 and M_2 of H_2. Since both H_1 and H_2 are equimatchable by Proposition 2.2.2, the matchings M_1 and M_2 can be extended to maximum matchings M_1' of H_1 and M_2'

of H_2, respectively. Clearly, the matching M'_1 covers all vertices of H_1 and M'_2 covers all but one vertices of H_2. Therefore, the matching M' defined by $M' = M'_1 \cup M'_2$ is a maximum matching of G.

Case 2: either ux or vy is an edge of M, but not both. We can assume that ux is an edge of M and vy is not an edge of M. Let $H'_1 = H_1 - u$ and $H'_2 = H_2 - x$. Observe that H'_1 is isomorphic with $K_{n,n-1}$ and H'_2 is isomorphic with $K_{m,m}$. Consequently, by Proposition 2.2.2 H'_1 is equimatchable and any its maximum matching misses exactly one vertex and H'_2 is equimatchable and has a perfect matching. The matching M is a disjoint union of matching M_1 of H'_1, matching M_2 of H'_2, and the edge ux. By equimatchability of H'_1 and H'_2 the matching M_1 extends to a matching M'_1 of H'_1 missing exactly one vertex and M_2 extends to a perfect matching of H'_2. The matching M' defined by $M' = M'_1 \cup M'_2 \cup \{ux\}$ is the desired matching of G missing exactly one vertex.

Case 3: both ux and vy are edges of M. Let $H'_1 = H_1 \setminus \{u, v\}$ and $H'_2 = H_2 \setminus \{x, y\}$. Observe that H'_1 is isomorphic with $K_{n-1,n-1}$ and H'_2 is isomorphic with $K_{m-1,m}$ and thus, by Proposition 2.2.2, both are equimatchable, H'_1 admitting a perfect matching and H'_2 a matching missing exactly one vertex. Clearly, M is a disjoint union of matchings M_1 of H'_1, M_2 of H'_2, and edges ux and vy. Again, M_1 extends to a perfect matching M'_1 of H'_1 and M_2 extends to a matching M'_2 of H'_2 missing exactly one vertex. Therefore, the matching M' defined by $M' = M'_1 \cup M'_2 \cup \{ux, vy\}$ is a matching of G missing exactly one vertex. □

Although we need the following lemma only for $K_{n,n}$ and $K_{m+1,m}$, we state it in a general form since the proof is identical.

Lemma 2.2.4. *Let a, b, c, d be positive integers such that $c > d$. Let u and v be two adjacent vertices of $K_{a,b}$. Then there are two distinct vertices x and y from the larger partite set of $K_{c,d}$ such that the graph G defined by $G = K_{a,b} \cup K_{c,d} \cup \{ux, vy\}$ has the genus equal to $\gamma(K_{a,b}) + \gamma(K_{c,d})$. Similarly, there are two distinct vertices $\tilde{x}$ and $\tilde{y}$ from the larger partite set of $K_{c,d}$ such that the graph G defined by $\tilde{G} = K_{a,b} \cup K_{c,d} \cup \{u\tilde{x}, v\tilde{y}\}$ has the genus equal to $\tilde{\gamma}(K_{a,b}) + \tilde{\gamma}(K_{c,d})$.*

Proof. We start by constructing the desired graph G and its embedding of genus $\gamma(K_{a,b}) + \gamma(K_{c,d})$. Denote by H_1 a copy of $K_{a,b}$ and by H_2 a copy of $K_{c,d}$. Let Π_i be a minimum-genus embedding of H_i for $i \in \{1, 2\}$. Since the vertices u and v are adjacent, there is a face F_1 of Π_1 such that both u and v lie on the boundary of F_1. Because H_2 is bipartite, any face of Π_2 has length at least four and thus contains at least two vertices from the larger partite set of H_2. Let x and y be arbitrary two vertices of the larger partite set of H_2 that lie together on the boundary of a face F_2 of Π_2 and let $G = H_1 \cup H_2 \cup \{ux, vy\}$. Adding one end of the edge ux into the interior of F_1 and the other end of ux into the interior of F_2 merges these faces into one face F, producing an embedding Π of connected graph $H_1 \cup H_2 \cup \{ux\}$ in the surface of genus $\gamma(H_1) + \gamma(H_2)$. Consequently, both v and y lie on the boundary of F and the edge vy can be added into Π without raising the genus, yielding the desired embedding of G in the surface of genus $\gamma(H_1) + \gamma(H_2)$. Since H_1 and H_2 are disjoint subgraphs of G, we get that $\gamma(G) \geq \gamma(H_1) + \gamma(H_2)$, which completes the proof of the orientable case. The proof of the nonorientable case is completely analogous. □

Theorem 2.2.5. *For any nonnegative integers g and h there exist 2-connected factor-critical equimatchable graphs G and $\tilde{G}$ such that G has orientable genus g and at least $4\lfloor\sqrt{2g}\rfloor + 5$ vertices and $\tilde{G}$ has nonorientable genus h and at least $4\lfloor\sqrt{h}\rfloor + 5$ vertices.*

Proof. Let n and m be maximum integers such that $K_{n,n}$ is embeddable in the orientable surface of genus $\lfloor g/2\rfloor$ and $K_{m+1,m}$ is embeddable in the orientable surface of genus $\lceil g/2\rceil$. Let u and v be two adjacent vertices of $K_{n,n}$. By Lemma 2.2.4 there are two vertices x and y of $K_{m+1,m}$ such that the graph G defined by $G = K_{n,n} \cup K_{m+1,m} \cup \{ux, vy\}$ is 2-connected with orientable genus $\gamma(K_{n,n}) + \gamma(K_{m+1,m}) = \lfloor g/2\rfloor + \lceil g/2\rceil = g$. By Lemma 2.2.3, the graph G is equimatchable and factor-critical.

To complete the proof it suffices to bound the number of vertices of G from below by calculating the value of n and m. First suppose that g is even. It is not difficult to verify that $n = \lfloor\sqrt{2g}\rfloor + 2$ and that $m = \lfloor(3 + \sqrt{8g+1})/2\rfloor$. Since $\lfloor 2\alpha\rfloor \geq 2\lfloor\alpha\rfloor \geq \lfloor 2\alpha\rfloor - 1$ holds for any positive real number α, we get that $K_{m+1,m}$ has $2m+1 \geq 3 + \lfloor\sqrt{8g+1}\rfloor \geq 3 + 2\lfloor\sqrt{2g}\rfloor$ vertices. Consequently, G has at least $4\lfloor\sqrt{2g}\rfloor + 7$ vertices. If g is odd, then $n = \lfloor\sqrt{2g-2}\rfloor + 2$ and $m = \lfloor(3 + \sqrt{8g+9})/2\rfloor$. Since $\lfloor\sqrt{2g-2}\rfloor \geq \lfloor\sqrt{2g}\rfloor - 1$ for any positive integer g, $K_{n,n}$ has $2(2 + \lfloor\sqrt{2g-2}\rfloor) \geq 2 + 2\lfloor\sqrt{2g}\rfloor$ vertices. Similarly as in the case of even g we get that $K_{m+1,m}$ has at least $3 + 2\lfloor\sqrt{2g}\rfloor$ vertices. Therefore, G has at least $4\lfloor\sqrt{2g}\rfloor + 5$ vertices, which completes the proof of the orientable case. The nonorientable case is analogous. □

The following four lemmas enable us to obtain upper bounds on the size of 2-connected equimatchable factor-critical graphs embeddable in a fixed surface.

Lemma 2.2.6. *If G is a randomly matchable graph embeddable in the orientable surface of genus g (nonorientable genus h), then $|V(G)| \leq 4 + 4\sqrt{g}$, respectively $|V(G)| \leq 4 + 2\sqrt{2h}$.*

Proof. If G is a complete graph embeddable in the orientable surface of genus g, then $|V(G)| \leq (7 + \sqrt{1 + 48g})/2$ by Theorem 1.3.3. If G is a complete regular bipartite embeddable in the orientable surface of genus g, then $|V(G)| \leq 4 + 4\sqrt{g}$ by Theorem 1.3.3. The inequality $(7 + \sqrt{1 + 48g})/2 \leq 4 + 4\sqrt{g}$, which holds for any $g \geq 0$, implies the result in the orientable case. The proof of the nonorientable case is analogous. □

Lemma 2.2.7. *If G has a cellular embedding in a surface S and more than*

$$\frac{6\chi(S)}{5-d}$$

vertices for some $d \geq 6$, then $\delta(G) \leq d$.

Proof. We prove the lemma by contradiction. Suppose that $\delta(G) \geq d+1$ and consider an embedding of G in the surface S. Denote by p, q, and r the number of vertices and edges of G and the number of faces of the embedding, respectively. As $\delta(G) \geq d+1$ we have $2q \geq (d+1)p$. Since G is a simple graph, $2q \geq 3r$ holds by Theorem 1.3.7. Substituting the expressions for p and r from the last two inequalities into Euler-Poincaré formula yields

$$\chi(S) = p - q + r \leq \frac{2q}{d+1} - q + \frac{2q}{3} = \frac{q(5-d)}{3(d+1)}.$$

Using $d \geq 6$ and $2q \geq (d+1)p$ we have

$$\frac{q(5-d)}{3(d+1)} \leq \frac{p(d+1)}{2} \cdot \frac{5-d}{3(d+1)}$$

and therefore

$$\chi(S) \leq \frac{p(5-d)}{6},$$

which contradicts the assumption of the lemma. □

Lemma 2.2.8. *Let G be a 2-connected, factor-critical equimatchable graph embeddable in the surface with orientable genus g, respectively nonorientable genus h. If G has a vertex of degree at most d, then $|V(G)| \le 5 + 2d + 4\sqrt{g}$, respectively $|V(G)| \le 5 + 2d + 2\sqrt{2h}$.*

Proof. Let v be a vertex of G with degree d in G and M_v a minimal matching that isolates v. By Lemma 2.1.1 M_v covers at most $2d$ vertices. Let $G' = G \setminus (V(M_v) \cup \{v\})$. By Theorem 2.1.3 G' has at most one component, this component is randomly matchable, and Lemma 2.2.6 yields that $|V(G')| \le 4 + 4\sqrt{g}$, respectively $|V(G')| \le 4 + 2\sqrt{2h}$. Hence G is a union of vertex v, matching M_v, and G', and in the orientable case we have

$$|V(G)| = |\{v\}| + |V(M_v)| + |V(G')| \le 1 + 2d + |V(G')| \le 1 + 2d + 4 + 4\sqrt{g} \le 5 + 2d + 4\sqrt{g}.$$

In the nonorientable case $|V(G)| \le 1 + 2d + |V(G')| \le 5 + 2d + 2\sqrt{2h}$, which completes the proof. □

Lemma 2.2.9. *For any $d \ge 6$ such that*

$$\frac{6\,(2-2g)}{5-d} \le 5 + 2d + 4\sqrt{g}, \quad \textit{respectively} \quad \frac{6\,(2-h)}{5-d_0} \le 5 + 2d_0 + 2\sqrt{2h},$$

the maximum size of a 2-connected factor-critical equimatchable graph embeddable in the surface with orientable genus G (nonorientable genus h) is at most $5 + 2d + 4\sqrt{g}$, respectively $5 + 2d + 2\sqrt{2h}$ vertices.

Proof. We prove the lemma by contradiction. Let d be an integer such that $d \ge 6$ and let G be a 2-connected factor-critical equimatchable graph embeddable in the orientable surface of genus g with $|V(G)| > 5 + 2d + 4\sqrt{g}$. By our assumption

$$|V(G)| > \frac{6(2-2g)}{5-d}$$

and thus by Lemma 2.2.7 G has a vertex with degree d' such that $d' \le d$. Consequently, by Lemma 2.2.8 G has at most $5 + 2d' + 4\sqrt{g} \le 5 + 2d + 4\sqrt{g}$ vertices, which is a contradiction. The nonorientable case is analogous. □

Theorem 2.2.10. *Let $m(g)$, respectively $\tilde{m}(h)$, denote the maximum number of vertices of a 2-connected factor-critical equimatchable graph embeddable in the orientable surface of genus g, respectively nonorientable surface of genus h. Then the following inequalities hold.*
i) If $g \leq 2$ and $h \leq 2$, then

$$4\sqrt{2g} + 1 \leq m(g) \leq 4\sqrt{g} + 17 \quad \text{and} \quad 4\sqrt{h} + 1 \leq \tilde{m}(h) \leq 2\sqrt{2h} + 17.$$

ii) If $g \geq 3$ and $h \geq 3$, then

$$4\sqrt{2g} + 1 \leq m(g) \leq c_g\sqrt{g} + 5 \quad \text{and} \quad 4\sqrt{h} + 1 \leq \tilde{m}(h) \leq \tilde{c}_h\sqrt{h} + 5,$$

where $c_g \leq 12$ and $\tilde{c}_h \leq 10$ are positive real constants such that the sequences $(c_g)_{g=3}^{\infty}$ and $(\tilde{c}_h)_{h=3}^{\infty}$ are decreasing, $\lim_{g\to\infty} c_g = 2\sqrt{7}+2 < 7.3$, and $\lim_{h\to\infty} \tilde{c}_h = \sqrt{2}\left(\sqrt{7}+1\right) < 5.2$.

Proof. The lower bounds follow from Theorem 2.2.5 and the inequality $\lfloor \alpha \rfloor > \alpha - 1$ which holds for any real number α. To prove the upper bounds, we distinguish two cases.

i) From Lemma 2.2.7 follows that if G has more than $12(g-1)$ vertices, then it has a vertex of degree at most 6, and hence by Lemma 2.2.8 at most $17 + 4\sqrt{g}$ vertices. The proof is concluded by observing that $17 + 4\sqrt{g} > 12(g-1)$ holds for any $g \leq 2$. The nonorientable case is analogous.

ii) We start by determining the smallest d such that $d \geq 6$ and

$$\frac{6(2-2g)}{5-d} \leq 5 + 2d + 4\sqrt{g} \tag{2.1}$$

for a fixed integer $g \geq 3$. Solving (2.1) for d we get that

$$d_g = \frac{5 - 4\sqrt{g} + \sqrt{112g + 120\sqrt{g} + 129}}{4}$$

is minimal such d and it is easy to verify that for $g \geq 3$ is indeed $d_g \geq 6$. Therefore, by Lemma 2.2.9 $m(g) \leq 5 + 2d_g + 4\sqrt{g}$. Clearly, for the sequence $(c_g)_{g=3}^{\infty}$ defined by

$$c_g = \frac{5 + 4\sqrt{g} + \sqrt{112g + 120\sqrt{g} + 129}}{2\sqrt{g}}$$

$m(g) \leq c_g\sqrt{g} + 5$ for every $g \geq 3$. It can be verified by standard methods that the sequence is decreasing and has the claimed limit, which completes the proof of the orientable case. The nonorientable case is analogous. □

In the investigation of 3-connected equimatchable graphs embeddable in a fixed surface Kawarabayashi and Plummer [15] proved that there is no such bipartite graph embeddable with face-width at least 3 at all. It is easy to see that there are arbitrarily large planar bipartite 2-connected equimatchable graphs.

Proposition 2.2.11. *For any positive integer k there is a planar 2-connected bipartite equimatchable graph with at least k vertices.*

Proof. Clearly, for any integer $k \geq 2$ the complete bipartite graph $K_{k,2}$ has the desired properties. □

The following theorem shows that there are infinitely-many 2-connected bipartite equimatchable graphs with any given genus and face-width.

Theorem 2.2.12. *For any positive integers n, g, and k there exists a 2-connected bipartite equimatchable graph G with at least n vertices, orientable genus g, and an embedding in S_g with face-width k. Similarly, for any positive integers n, h, and k there exists a 2-connected bipartite equimatchable graph $\tilde{G}$ with at least n vertices, nonorientable genus h, and an embedding in N_h with face-width k.*

Proof. We prove only the orientable case, since the nonorientable case is analogous. Take a 2-connected graph G' with at least n vertices, genus g, and with a genus embedding Π' with face-width k, for example any sufficiently large 2-connected triangulation with a given genus and face-width; it is well known that such triangulations exist. We construct the desired graph G starting from G' by replacing every edge e of G' by l parallel edges $e_1, \dots, e_l$ for some fixed $l \geq 2$ and subdividing every edge e_i by a new vertex y_{e_i}. Denote by B the set of all vertices y_{e_i} of G, that is, $B = \{y_{e_i}; e \in E(G'), 1 \leq i \leq l\}$. Let $A = V(G) \setminus B$.

Clearly, G is bipartite and the vertices of A form the smaller partite set of G. By [19, Theorem 3] a connected bipartite graph is equimatchable if and only if for any vertex u from the smaller partite set there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| \leq |X|$. We prove that G is equimatchable by exhibiting such set X_v for every vertex v from A. If a vertex v is in G' incident with an edge $e = uv$, then let $X_v = \{y_{e_1}, \dots y_{e_l}\}$. In G we have $X_v \subseteq N(v)$ and $N(X_v) = \{u, v\}$, with possibly $u = v$ if uv is a loop. Since $l \geq 2$, we have $|N(X_v)| \leq |X_v|$. Therefore, for every vertex v from A there exists a non-empty set X_v such that $X_v \subseteq N(v)$ and $|N(X_v)| \leq |X_v|$, and hence by [19] H is equimatchable. It is easy to see that multiplying and subdividing edges does not change the genus of the graph, and thus $\gamma(G) = \gamma(H)$. To construct the desired genus embedding Π of G with face-width k, start with Π'. For any edge $e = uv$ of G', choose the preferred direction of e. If the preferred direction of e is from u to v, then in the rotation at u replace the occurrence of e by $e_1 \dots e_k$ and replace the occurrence of e^{-1} in the rotation at v by $e_k \dots e_1$. Finally, subdivide every edge e_i by the new vertex y_{e_i}. Clearly, the subdivided edges $e_1, \dots, e_k$ bound $l - 1$ faces of length 4. Moreover, the occurrence of $e = uv$ in its face boundary is replaced by a sequence of two edges $(uy_{e_1})(y_{e_1}v)$ and the occurrence of e^{-1} in its faces boundary is replaced by $(vy_{e_k})(y_{e_k}u)$. It is not difficult to see that union of any m faces of Π is union of at most m faces of Π' and hence the face-width of Π is at least k. Since in Π' there is a noncontractible curve of minimum length that intersects only vertices of G' (see [34]), there is a homotopically equivalent noncontractible curve whose intersection with G consists from precisely k vertices of G. Thus face-width of Π is at most k, which completes the proof. □

Theorem 2.2.12 and the results of [15] suggest the following open problem.

Problem. *Are there infinitely-many* 3*-connected bipartite equimatchable graphs embeddable in a given surface with face-width at most* 2*?*

3

Minimum cuts in equimatchable factor-critical graphs

The aim of this chapter is to describe the structure of equimatchable factor-critical graphs with respect to their minimum cuts, thus extending the results of Favaron [8] to graphs with higher connectivity.

Our main results can be described as follows. Let G be a k-connected equimatchable factor-critical graph with a k-cut S. If $|V(G)| \geq 2k+3$ or if $G-S$ has a component with at least k vertices, then $G-S$ has exactly two components which are close to complete or complete bipartite graphs. Moreover, if both components of $G-S$ have at least 3 vertices, then both are complete and, if additionally $k \geq 4$, then G has independence number 2. This implies that for $k \geq 4$ a k-connected graph with odd number of vertices and a k-cut S such that $G-S$ has two components with size at least 3 is equimatchable and factor-critical if and only if it has independence number 2.

Concerning the methods used in this chapter, we repeatedly use Theorem 2.1.3.

The chapter is organized as follows. Section 3.1 is devoted to the structure of equimatchable factor-critical graphs with respect to their minimum vertex cuts. In Section 3.2 we use structural results from Section 3.1 to show that an odd k-connected graph with $k \geq 4$, at least $2k+3$ vertices, and a k-cut which separates at least two components with at least 3 is equimatchable factor-critical if and only if it has independence number 2.

3.1 Vertex cuts in equimatchable factor-critical graphs

The aim of this section is to describe the structure of equimatchable factor-critical graphs with respect to their minimum vertex cuts. Favaron [8] provided a complete characterisation of equimatchable factor-critical with connectivity 1 and 2 with respect to their minimum vertex cuts. While the 1-connected case is somewhat exceptional, our results for connectivity $k \geq 3$ are in nature very similar to Theorem 3.1.2. In particular, the difficulties with describing the whole larger component in the case when the smaller component is a singleton carries completely to large connectivity, as can be seen from Theorem 3.1.5.

Theorem 3.1.1 (Favaron [8]). *A graph G with vertex-connectivity 1 is equimatchable and factor-critical if and only if all of the following conditions hold:*
(1) G has exactly one cut-vertex d;
(2) every connected component C_i of $G - d$ is isomorphic to K_{2r_i} or to K_{r_i,r_i} for some integer r_i.
(3) d is adjacent to at least two adjacent vertices of each C_i.

Theorem 3.1.2 (Favaron [8]). *Let G be a 2-connected equimatchable factor-critical graph with at least 4 vertices and a two cut $S = \{s_1, s_2\}$. Then $G \setminus S$ has precisely two components, one of them even and the other odd. Let A and B denote the even, respectively the odd component of $G \setminus S$, let $a_1, a_2 \in A$ be two distinct vertices adjacent to s_1 and s_2, respectively, and, if $|B| > 1$, let b_1, $b_2 \in B$ be two distinct vertices adjacent to s_1 and s_2 respectively. Then G has the following structure:*
(1) B is one of the four graphs K_{2p+1}, $K_{2p+1} \setminus \{b_1b_2\}$, $K_{p,p+1}$, $K_{p,p+1} \cup \{b_1b_2\}$ for some nonnegative integer p. In the two last cases all neighbours of S in B belong to the $(p+1)$-stable set of $K_{p,p+1}$.
(2) $A \setminus \{a_1, a_2\})$ is either K_{2q-2} or $K_{q-1,q-1}$ for some nonnegative integer q and, if $|B| > 1$, then A is either K_{2q}, or $K_{q,q}$.

We are able to extend these results to arbitrary fixed vertex connectivity k by showing that if the graph has at least $2k + 3$ vertices, then there are exactly two components and both are almost complete or complete bipartite. We start with a lemma which allows us to efficiently apply Theorem 2.1.3 to bound the number of components.

Lemma 3.1.3. *Let G be a factor-critical equimatchable graph, M a matching of G,* *and H an odd component of $G \setminus V(M)$. Then $G \setminus (H \cup V(M))$ is isomorphic with K_{2n} or $K_{n,n}$ for some n.*

Proof. Since G is equimatchable, the matching M can be extended to a maximum matching M' of G. The fact that G is factor-critical implies that M' leaves uncovered exactly one vertex v of G. Clearly, M' cannot cover all vertices of H and hence v lies in H. The matching M' covers all neighbours of v and thus it is an isolating matching of v. Consider any minimal matching M_v such that $M_v \subseteq M'$ and M_v isolates v. Let G' denote the graph $G \setminus (V(M_v) \cup \{v\})$. By Theorem 2.1.3 the graph G' is K_{2n} or $K_{n,n}$ for some n. It is not difficult to see that M_v can contain only edges of M and edges of H, and thus $V(H) \cup V(M_v) \subseteq V(H) \cup M'$. It follows that the graph $G \setminus (H \cup V(M))$ can be obtained from G' by removing the vertices covered by $M \setminus M_v$. It is easy to see that removing two vertices joined by an edge from K_{2n} or $K_{n,n}$ leads to K_{2n-2}, respectively to $K_{n-1,n-1}$. We conclude that $G \setminus (H \cup V(M))$ is isomorphic with K_{2m} or $K_{m,m}$ for some m, as claimed. □

The next lemma guarantees the existence of a large number of independent edges between a component and any subset of the cut.

Lemma 3.1.4. *Let G be a graph with vertex connectivity k and with a k-vertex-cut S. Let H be a component of $G \setminus S$. Then for arbitrary set of vertices $X \subseteq S$ the graph G contains at least* $\min(|H|, |X|)$ *independent edges between H and X.*

Proof. We prove the lemma by contradiction. Let l be the maximum number of independent edges in G between H and X and suppose that $l < \min(|H|, |X|)$. Since any set of independent edges between H and X is a matching between vertices of H and X, any maximum matching between H and X has size l. By König's theorem [17] the maximum size of a matching between H and X equals the minimum cardinality of a vertex cover of all edges between H and X. Hence there is a vertex set $Y \subseteq (H \cup X)$ such that $|Y| = l$ and Y cover all edges between H and X. Since $|Y| < |H|$, the set $H \setminus Y$ contains at least one vertex and $(S \setminus X) \cup Y$ is a vertex cut of G. Using $|Y| < |X|$ we get that the size of $(S \setminus X) \cup Y$ satisfies $(|S| - |X|) + |Y| = k - |X| + |Y| < k$, which contradicts the fact that G is k-connected. □

In the case when the smaller component of $G - S$ has precisely one vertex we are now ready to prove that the larger component, except the vertices matched with the cut, is complete or complete bipartite. However, as stated earlier, a description of the structure of the graph induced on $V(M)$ and of the edges between $V(M)$ and C seems to be quite difficult and is left as an open problem.

Theorem 3.1.5. *Let G be a k-connected equimatchable factor-critical graph with a k-cut S. Assume that $G \backslash S$ has a component C with at least k vertices and $G \backslash (S \cup C)$ has a component consisting from a single vertex. Then $G - S$ has exactly two components and there is a matching M between S and C covering all vertices of S. Furthermore, $C - V(M)$ is isomorphic with K_{2n} or $K_{n,n}$ for some integer n.*

Proof. The existence of a matching M between S and C covering all vertices of S is a consequence of Lemma 3.1.4. Let v be the vertex of a single-vertex component of $G - S$. Lemma 3.1.4 implies that v is adjacent to every vertex of S and thus M is a minimal isolating matching of v. By Theorem 2.1.3 the graph $G \setminus (V(M) \cup \{v\})$ is connected and isomorphic with K_{2n} or $K_{n,n}$ for some integer n, which completes the proof. □

The next lemma guarantees that if the graph has at least $2k+3$ vertices, then removing any minimum cut yields precisely two components.

Lemma 3.1.6. *Let $k \geq 3$ and G be a k-connected factor-critical equimatchable graph with at least $2k + 3$ vertices. Then $G - S$ has precisely two components.*

Proof. We prove the lemma by contradiction and suppose that $H_1, \ldots, H_l$ are components of $G \backslash S$ for some $l \geq 3$. Let M be a matching between S and $H_1 \cup H_2$ which leaves uncovered odd number of vertices of both H_1 and H_2 and covers as many vertices of S as possible. Observe that such a matching always exists since $k \geq 3$ and, by Lemma 3.1.4, every vertex of S is adjacent to every component of $G - S$. First we prove that if M leaves uncovered at least 2 vertices of S, then it leaves uncovered precisely one vertex in both H_1 and H_2. Indeed, suppose for the contrary that M leaves uncovered at least two vertices s_1 and s_2 of S and more than one vertex in, say, H_1. Denote by M_1 the edges of M incident with H_1. Let $X = \{s_1, s_2\} \cup (S \cap V(M_1))$. Since M leaves uncovered at least 3 vertices of H_1, applying Lemma 3.1.4 to H_1 and X yields that there is a matching M' between H_1 and X.

It can be easily seen that $M'' = M' \cup (M \setminus M_1)$ is a matching between S and $H_1 \cup H_2$ which leaves uncovered odd number of vertices in both H_1 and H_2, and that M'' is larger than M, which contradicts the maximality of M.

We proceed to extend M to a matching N covering all vertices of S and leaving uncovered odd number of vertices in both H_1 and H_2. If M covers all vertices of S, then let $N = M$. If M leaves uncovered precisely one vertex s of S, then let $N = M \cup \{e\}$, where e is any edge joining s with H_3, note that such an edge always exists by Lemma 3.1.4. Finally, if M leaves uncovered at least 2 vertices of S, then it leaves uncovered exactly one vertex in both H_1 and H_2 as shown above, and $|V(G)| \geq 2k+3$ implies that $H_3 \cup \cdots \cup H_l$ contains more vertices than $S - V(M)$. Therefore, by Lemma 3.1.4 there is a matching N' between $S - V(M)$ and $H_3 \cup \cdots \cup H_l$ covering all vertices of $S - V(M)$. Now $N = M \cup N'$ is the desired matching covering all vertices of S and leaving uncovered odd number of vertices in both H_1 and H_2.

To complete the proof we show that N cannot be extended to a maximum matching of G, contradicting the fact that G is equimatchable. Indeed, N leaves uncovered odd number of vertices of both H_1 and H_2 and separates H_1 and H_2 from the rest of the graph and thus any maximal matching $N'' \supseteq N$ leaves uncovered at least one vertex of both H_1 and H_2. Since G is equimatchable and factor-critical, any maximum matching of G leaves uncovered precisely one vertex of G and hence N'' cannot be a maximum matching. The proof is now complete. □

To deal with the cases when the smaller component of $G - S$ has at least two vertices we will need the following lemma.

Lemma 3.1.7. *Let G be a k-connected equimatchable factor-critical graph with a k-cut S. Assume that $G \setminus S$ has a component C with at least k vertices and $G \setminus (S \cup C)$ has a component with exactly two vertices. Then $G - S$ has exactly two components and there is a matching M between S and C covering all vertices of S. Furthermore, for every vertex $x \in C \cap V(M)$, the subgraph of G induced by $C - V(M) \cup \{x\}$ is isomorphic with K_{2n} or $K_{n,n}$ for some integer n.*

Proof. The existence of a matching M between S and C covering all vertices of S is a consequence of Lemma 3.1.4. Let D be a component of $G \setminus (C \cup S)$ with exactly two vertices. Let s be the vertex of S matched with x. Lemma 3.1.4 implies that there is a vertex of D, say d, adjacent to s. Let d' be the vertex of D different from d. Consider the set $M' = M \setminus \{sx\} \cup \{ds\}$; clearly $\{d'\}$ is an odd component of $G \setminus V(M')$. Thus by Lemma 3.1.3 the graph $G \setminus (\{d'\} \cup V(M')) = C - V(M) \cup \{x\}$ is isomorphic with K_{2n} or $K_{n,n}$ for some integer n, which completes the proof. □

The following theorem provides a complete characterisation of k-connected equimatchable factor-critical graphs with a k-cut S such that $G - S$ contains a component with at least k vertices and a component with precisely 2 vertices.

Theorem 3.1.8. *Let G be a k-connected equimatchable factor-critical graph with a k-cut S. Assume that $G \setminus S$ has a component C with at least k vertices and $G \setminus (S \cup C)$ has a component with exactly two vertices. Then $G - S$ has exactly two component. Furthermore, if S contains an edge, then C is a complete graph. If S does not contain an edge, then then there is a nonegative integer m and sets $\{x_1, \ldots, x_m\}$ of vertices of C and $\{y_1, \ldots, y_m\}$ of vertices of S such that x_iy_i is not an edge of G for every $i \in \{1, \ldots, k\}$ and $C \cup S \cup \{x_1y_1, \ldots, x_ky_k\}$ is isomorphic with $K_{n,n+1}$ for some n.*

Proof. Let $S = \{s_1, \ldots, s_k\}$ and let $D = \{d_1, d_2\}$ be a component of $G \setminus (C \cup S)$ with exactly two vertices. Note that D is the only component of $G \setminus (C \cup S)$ by Lemma 3.1.6. By Lemma 3.1.4 there is a matching M between S and C which covers all vertices of S. For every $i \in \{1, \ldots, k\}$ we denote by c_i the vertex of C joined to s_i by M and let $X = \{c_1, \ldots, c_k\}$ and $C' = C \setminus X$. By Lemma 3.1.4 for every $i \in \{1, \ldots, k\}$ there is j such that $j \in \{1, 2\}$ and s_i is adjacent to d_j. Clearly, the set M_i defined by $M_i = (M \setminus \{s_ic_i\}) \cup \{s_id_j\}$ is a matching isolating d_{3-j}. Therefore, by Theorem 2.1.3 the graph $G \setminus (V(M_i) \cup \{d_{3-l}\}) = C' \cup \{c_i\}$ is isomorphic with K_{2n} or $K_{n,n}$ for some n. In particular, the graph $G \setminus S$ has exactly two components.

Claim. If there is edge in X, then there is edge in S.

Proof of Claim. Let the edge in X be c_1c_2. Let M' be arbitrary matching of $C' \cup \{c_3\}$. Matching $M'' = (M \setminus \{s_1c_1, s_2c_2, s_3c_3\}) \cup M' \cup \{d_1d_2, c_1c_2\}$ leaves vertices s_1, s_2, and s_3 unmatched. Hence there is edge in $\{s_1, s_2, s_3\}$ and the claim follows.

The rest of the proof is split into two cases; the above Claim implies that these cases cover all possibilities.

Case a) There is an edge in S.

Claim 1. If rs is an edge in S and u and v are the two vertices of C matched by M with r and s, respectively, then $\{u, v, w\}$ is a triangle for any vertex w in X.

Proof of Claim 1. Choose arbitrary vertex w from X and let s be the vertex of S joined to w by M. By Lemma 3.1.4 there is an edge between s and a vertex of D, say d. Let $M' = (M \setminus \{ru, sv, tw\}) \cup \{rs, td\}$ and denote the only vertex of $D - \{d\}$ by d'. Clearly, M' is a matching of G which isolates d'. For any $x \in \{u, v, w\}$ there is a perfect matching M_x of $C' \cup \{x\}$ by Theorem 3.1.8. For any $x \in \{u, v, w\}$, the set $M_x \cup M'$ is a matching of G which leaves uncovered only the vertices $\{d', u, v, w\} - \{x\}$. Since S is a cut separating D and C, there is no edge between d' and $\{u, v, w\}$. Therefore, the fact that G is equimatchable and factor-critical implies that the two vertices in $\{u, v, w\} - \{x\}$ are joined by an edge, which completes the proof of the claim.

Claim 2. The subgraph of G induced by X is a complete graph.

Proof of Claim 2. If $k = 3$, then the claim follows directly from Claim 1. Therefore, assume that $k \geq 4$ and that the edge in S is rs. Our aim is to show that there is an edge between two arbitrary vertices x and y of X. Let x and y be two vertices of X and denote by c_r and c_s the two vertices of X joined by M to r and s, respectively. If x or y belongs to $\{c_r, c_s\}$, then x and y are joined by an edge by Claim 1. Hence we can assume that $\{x, y\} \cap \{c_r, c_s\} = \emptyset$. Claim 1 applied to $\{c_r, c_s, x\}$ shows that $c_r x$ is an edge of G. By Theorem 3.1.8 the graph $C' \cup \{y\}$ is isomorphic with K_{2n} or $K_{n,n}$ for some n. Let M' be a perfect matching of $C' \cup \{y\}$ and let s_x and s_y be the vertices of S joined by M to x and y, respectively. Denote the vertices of D by d_1 and d_2 and consider the set $M'' = (M \setminus \{rc_r, xs_x, ys_y\}) \cup M' \cup \{xc_r, d_1 d_2\}$. It is not difficult to see that M'' is a matching which leaves uncovered exactly the vertices r, s_x, and s_y. Hence $\{r, s_x, s_y\}$ contains an edge e and the result follows by using Claim 1 on e with $\{c_r, x, y\}$ in the role of $\{u, v, w\}$.

Claim 3. The subgraph of G induced by C' is a complete graph.

Proof of Claim 3. Assume that the edge in S is rs. By Theorem 3.1.8 the graph C' is isomorphic with either K_n or $K_{n,n+1}$ for som integer n. Let c_r and c_s be the vertices of C joined by M to r and s, respectively, and let x be an arbitrary vertex of $X \setminus \{c_r, c_s\}$. By Claim 1 applied to rs the subgraph of G induced by $\{x, c_r, c_s\}$ is a triangle. Let s_x be the vertex of S joined to x by M. By Lemma 3.1.4 there is a vertex of D, say d, adjacent to s_x. Let d' be the vertex of D different from d. Consider the set $M' = M \setminus \{rc_r, sc_s, xs_x\} \cup \{rs, ds_x\}$; clearly M' is a matching isolating d'. By Theorem 2.1.3 the graph $G \setminus (V(M) \cup \{d'\}) = C' \cup \{x, c_r, c_s\}$ is either K_{2n} or $K_{n,n}$.

Since $C' \cup \{x, c_r, c_s\}$ contains $\{x, c_r, c_s\}$ which induce a triangle, it is isomorphic with a complete graph and hence C' is a complete graph, as claimed.

Claim 4. The subgraph of G induced by C is a complete graph.

Proof of Claim 4. By Claim 2 the set X induces a complete graph and by Claim 3 the set C' induces a complete graph. Lemma 3.1.7 implies that if C' is complete, then every vertex of X is adjacent to every vertex of C' and hence C is a complete graph.

The preceding claim completes the case when there is an edge in S and the first part of the proof.

Case b) Both X and S are independent sets.

Claim 1. The subgraph of G induced by C' is isomorphic with $K_{n,n+1}$ for some $n \geq 0$.

Proof of Claim 1. If C' contains only one vertex, then the claim holds. Since the number of vertices of C' is odd, we can assume that $|V(C')| \geq 3$. For every $i \in \{1, \dots, k\}$ Lemma 3.1.7 implies that $C' \cup \{c_i\}$ is isomorphic with K_{2m} or $K_{m,m}$ for some m. If $C' \cup \{x_i\}$ is $K_{m,m}$ for some $i \in \{1, \dots, k\}$, then C' is clearly $K_{m-1,m}$ and the claim holds. We proceed by contradiction to show that $C' \cup \{c_1\}$ cannot be an even complete graph. Suppose that $C' \cup \{c_1\}$ is an even complete graph, and let M' be its perfect matching and xy an edge of M'. Clearly, the set $(M \setminus \{s_1c_1, s_2c_2, s_3c_3\}) \cup (M' \setminus \{xy\}) \cup \{c_2x, c_3y, d_1d_2\}$ is a matching which covers all vertices of G except s_1, s_2, and s_3. The facts that S is an independent set and that G is equimatchable and factor-critical yields a contradiction, which completes the proof of the claim.

Denote by U the smaller and by W the larger partite set of C'.

Claim 2. There is no edge between X and U.

Proof of Claim 2. We proceed by contradiction. Suppose that u is a vertex of U adjacent to a vertex x in X. Let s be the vertex of S joined to x by M and let d be a vertex of D joined to s; such a vertex d exists by Lemma 3.1.4. Clearly, the set $M' = (M \setminus \{sx\}) \cup \{ds, xu\}$ is a matching of G. It is not difficult to see that any maximal matching containing M' leaves unmatched at least two vertices of W, which contradicts the fact that G is equimatchable and factor-critical.

Claim 3. There is no edge between S and W.

Proof of Claim 3. For a contradiction suppose that there is a vertex w of W incident with s_i for some $i \in \{1, \ldots, k\}$. Let s be a vertex of S incident with a vertex d of D and different from s_i; such a vertex s exists by Lemma 3.1.4. Let $N = (M \setminus \{s_i x_i, s_j x_j\}) \cup \{s_i c, s_j d\}$. Clearly, N is a matching of G. It is not difficult to see that any maximal matching of G containing N leaves at least two vertices of $U \cup \{x_i, x_j\}$ unmatched, which contradicts the fact that G is factor-critical and equimatchable.

Claim 4. Every vertex of X is adjacent with every vertex of W.

Proof of Claim 4. Let x be a vertex of X and w a vertex of W. By Claim 1 the graph C' is $K_{n,n+1}$ and by its choice w lies in the larger partite set of C' It follows that there is a perfect matching M' of $C' - \{w\}$. Let s be the vertex of S matched with x by M. By Lemma 3.1.4 there is an edge e between s and D. Let d be the vertex of D not covered by e. Let $M'' = (M - \{xs\}) \cup M' \cup \{e\}$. Clearly, M'' is a matching which covers all vertices of G except d, x and w. Since x and w lie in C and C and D are different components of $G - S$, the vertex d is adjacent with neither x, nor w. Using the fact that G is factor-critical and equimatchable we get that x and w are adjacent, which completes the proof.

Claim 5. Any vertex s of S is incident with either all, or all but one vertices of $X \cup U$.

Proof of Claim 5. Suppose for the contrary that there is a vertex s of S and two vertices x and y from $X \cup U$ such that s is incident neither with x, nor with y. Let v be the vertex of X matched with s by M. Let z_1 and z_2 be two different edges of $X - \{x\}$. (Note that we do not require $\{z_1, z_2\} \cap \{x, y\} = \emptyset$.) Denote by t_1 and t_2 the two vertices of S adjacent to z_1 and z_2, respectively. Let M' be a set of two independent edges between D and $\{t_1, t_2\}$; such two edges exist by Lemma 3.1.4. By Claim 1 the graph C' is isomorphic with $K_{n,n+1}$ and by Claim 4 every vertex of X is adjacent with every vertex of W. It follows that $C' \cup \{v, z_1, z_2\}$ has a matching M'' which covers all vertices except x and y. Consider the set $M''' = (M \setminus \{sv, z_1 t_1, z_2 t_2\}) \cup M' \cup M''$. It is not difficult to see that M''' is a matching which covers all vertices of G except x, y and s. Observe that there is no edge between x and y. Indeed, if one of x, y belong to X and the other to U, then they are not adjacent by Claim 2. If both x and y are from X, then they are not adjacent by the assumption of Case b). Finally, if both x and y are from U, then they are not adjacent by the definition of U. Since by our assumption s is adjacent with neither x, nor y, we get a contradiction with the fact that G is factor-critical and equimatchable.

Claim 6. Any vertex x of X is incident with either all, or all but one vertices of S.

Proof of Claim 6. Suppose for the contrary that there is a vertex x of X and two vertices t_1 and t_2 of S such that x is incident with neither t_1 nor t_2. Let s be the vertex of S matched with x by M and let y_1 and y_2 be the two vertices matched by M with t_1, respectively t_2. By Claim 5 the vertex s is adjacent with either y_1, or y_2; without loss of generality we can assume that s is incident with y_1. By Claim 1 the graph C' is isomorphic with $K_{n,n+1}$ and by Claim 4 the vertex y_2 is incident with every vertex form the larger partite set of C'. Therefore, there exists a perfect matching M'' of $C' \cup \{y_2\}$. Let $M'' = (M \setminus \{sx, t_1y_1, t_2y_2\}) \cup M' \cup \{e, sy_1\}$, where e is the edge in D. It is not difficult to see that M'' is a matching which covers all vertices of G except x, t_1, and t_2. By the assumption of Case b) the vertices t_1 and t_2 are not adjacent and by our assumption x is incident with neither t_1, nor t_2, which contradicts the fact that G is factor-critical and equimatchable.

Claim 7. Any vertex u of U is incident with either all, or all but one vertices of S.

Proof of Claim 7. Suppose for the contrary that there is a vertex u of U and two vertices t_1 and t_2 of S such that u is incident with neither t_1 nor t_2. Let y_1 and y_2 be the two vertices of X matched by M with t_1, respectively t_2. By Claim 1 the graph C' is isomorphic with $K_{n,n+1}$ and by Claim 4 both vertices y_1 and y_2 are incident with every vertex form the larger partite set of C'. Therefore, there exists a matching M' of $C' \cup \{y_1, y_2\}$ covering all vertices except u. Let $M'' = (M \setminus \{t_1y_1, t_2y_2\}) \cup M' \cup \{e\}$, where e is the edge in D. It is not difficult to see that M'' is a matching which covers all vertices of G except u, t_1, and t_2. By the assumption of Case 2. the vertices t_1 and t_2 are not adjacent and by our assumption u is incident with neither t_1, nor t_2, which contradicts the fact that G is factor-critical and equimatchable.

Denote the subgraph of G induced by $C \cup S$ by H. Claim 2 and 3 imply that $U \cup W \cup X \cup S = H$ is a bipartite graph with partite sets $X \cup U$ and $S \cup W$. Claim 1 and the definition of U and W yields that every vertex of U is adjacent to every vertex of W. By Claim 4 every vertex of X is incident with every vertex of W. From Claim 5, 6, and 7 we get that there is a nonnegative integer m and vertices $t_1, \dots, t_m$ and $y_1, \dots, y_m$ such that $t_iy_i \notin E(H)$ for all $i \in \{1, \dots, m\}$ and that $H \cup \{t_1y_1, \dots, t_my_m\}$ is a complete bipartite graph. The proof is now complete. □

Theorem 3.1.9. *Let G be a k-connected equimatchable factor-critical graph with at least $2k+3$ vertices and a k-vertex-cut S, where $k \geq 3$. If $G - S$ has two components with at least 3 vertices, then $G - S$ has exactly two components and both are complete graphs.*

Proof. By lemma 3.1.6 the graph $G - S$ has precisely two components, denote these components by C and D, respectively. Furthermore, denote the vertices of S by $s_1, \ldots, s_k$. First we deal with the case when both C and D are strictly smaller than k; this case is much simpler. Take any two vertices x and y of a component of $G - S$, say of C. Let $l = |V(C)|$. Since $|V(C)| < k$, there are l independent edges between S and C by Lemma 3.1.4. Therefore, we can choose a set M_1 of $l-2$ independent edges between S and $C - \{x, y\}$. Since $|V(G)| \geq 2k+3$, there is a set M_2 of $k - l + 2$ independent edges between D and $S - V(M_1)$. Let $M = M_1 \cup M_2$; observe that M is a matching of G. It is not difficult to see that the vertex x can be in $G - V(M)$ adjacent only to y, and similarly y can be adjacent only to x. Since G is equimatchable and factor-critical, the matching M can be extended to a maximum matching of G, which necessarily leaves unmatched precisely one vertex of G. Clearly, this is possible only if x and y are adjacent. Since the choice of x and y was arbitrary, it follows that both components of $G - S$ are complete, as claimed.

From now on we assume that at least one component of $G - S$, say C, has at least k vertices. By Lemma 3.1.4 there is a set M of k independent edges between C and S covering all vertices of S. Clearly, M is a matching of G. Denote the edges of M by $\{e_1, \ldots, e_k\}$, where $e_i = s_i u_i$ and $u_i \in C$ for all $i \in \{1, \ldots, k\}$. We distinguish two cases.

Case a) $|C| - k$ is odd. Let X be an odd component of $C \setminus \{u_1, \ldots, u_k\}$. Clearly, G, M, and X satisfy the assumptions of Lemma 3.1.3 which implies that $G \setminus (X \cup M)$ is isomorphic with K_{2n} or $K_{n,n}$ for some n. It follows that X is the only component of $C \setminus \{u_1, \ldots, u_k\}$ and $X \cup M = S \cup C$. Consequently, $D = G \setminus (X \cup M)$ and hence D is isomorphic with K_{2n} or $K_{n,n}$ for some n. To prove that D is isomorphic with K_n we proceed by contradiction. Suppose that D is not a complete graph and thus $|D| \geq 4$. Since $k \geq 3$, by Lemma 3.1.4 there are at least three independent edges between D and S and at least two of these edges, say f_1 and f_2, have their endvertices in the same partite set of D. Without loss of generality assume that $f_1 = s_1 w_1$ and $f_2 = s_2 w_2$, where w_1 and w_2 are vertices of D. Let $M' = (M \setminus \{e_1, e_2\}) \cup \{f_1, f_2\}$ and let X' be an odd component of $C \setminus \{u_3, \dots, u_k\}$. By Lemma 3.1.3 the graph $G \setminus (X' \cup M')$ is isomorphic with K_{2n} or $K_{n,n}$ for some n. On the other hand, $G \setminus (X' \cup M') = D \setminus \{w_1, w_2\}$, which contradicts the fact that D is a complete bipartite graph with w_1 and w_2 lying in the same partite set. Therefore, we conclude that D is isomorphic with K_{2n}.

Let y_1, y_2, and y_3 be arbitrary three vertices from $\{u_1, \dots, u_k\}$ and let x_i be the vertex joined with y_i by M for all $i \in \{1, 2, 3\}$. By Lemma 3.1.4 there are three pairwise independent edges f_1, f_2, and f_3 between D and $\{x_1, x_2, x_3\}$; without loss of generality we can assume that $f_i = x_i z_i$ for every $i \in \{1, 2, 3\}$, where z_1, z_2, and z_3 are vertices of D. Denote the set $C \setminus \{u_1, \dots, u_k\}$ by C'. For every $i \in \{1, 2, 3\}$ let $D_i = D \setminus \{z_i\}$, $S_i = S \setminus \{x_i\}$, $C'_i = C' \cup \{y_i\}$, and $M_i = (M \setminus \{x_i y_i\}) \cup \{f_i\}$. Similarly, let $D_4 = D \setminus \{z_1, z_2, z_3\}$, $S_4 = S \setminus \{x_1, x_2, x_3\}$, $C'_4 = C' \cup \{y_1, y_2, y_3\}$, and $M_4 = (M \setminus \{x_1 y_1, x_2 y_2, x_3 y_3\}) \cup \{f_1, f_2, f_3\}$. It is easy to see that the graph D_4 is an odd component of $G \setminus V(M)$ and hence C'_4 is isomorphic with $K_{m,m}$ or K_{2m} for some integer m by Lemma 3.1.3.

We proceed by contradiction to show that C'_4 is isomorphic with a complete graph. To prove this, suppose that C'_4 is isomorphic with $K_{m,m}$ for some integer $m \geq 2$. For every $i \in \{1, 2, 3\}$ the graph D_i is an odd component of $G \setminus V(M)$ and hence C'_i is isomorphic with a complete or complete bipartite graph by Lemma 3.1.3. Since C'_4 is $K_{m,m}$, the graph D_i is isomorphic with $K_{m-1,m-1}$ for every $i \in \{1, 2, 3\}$. Comparing C'_4 and C'_1 yields that y_2 and y_3 lie in different partite sets of C'_4. Similarly, comparing C'_4 and C'_2 yields that y_1 and y_3 lie in the different partite sets of C'_4. Finally, comparing C'_4 and C'_3 yields that y_1 and y_2 lies in different partite sets of C'_4. Since C'_4 is a complete bipartite graph, such a distribution of y_1, y_2 and y_3 is not possible, which is a contradiction. Consequently, C'_4 is isomorphic with a complete graph. Since the choice of y_1, y_2 and y_3 was arbitrary, it follows that C is isomorphic with a complete graph, as claimed.

Case b) $|C| - k$ is even. First observe that in this case D has odd number of vertices. Since C and D are the only components of $G - S$ and G has at least $2k + 3$ vertices, we get that D has at least 3 vertices. Let $l = \min\{|D|, k\}$, where $l \geq 3$. By Lemma 3.1.4 there is a set of l independent edges f_i, $i \in \{i, \dots, l\}$, between D and S. Without loss of generality we can assume that $f_i = s_i w_i$ for $i \in \{1, \dots, l\}$, where $\{w_1, \dots, w_l\}$ are distinct vertices of D. For every $i \in \{1, \dots, l\}$ let X_i be an odd component of $(C \setminus \{c_1, \dots, c_k\}) \cup \{c_i\}$, let $M_i = (M \setminus \{e_i\}) \cup \{f_i\}$, and let $D_i = D - \{w_i\}$. For every $i \in \{1, \dots, l\}$ Lemma 3.1.3 applied to G, M_i, and X_i yields that $G \setminus (X_i \cup V(M_i))$ is isomorphic with K_{2n} or $K_{n,n}$ for some n and thus X_i is the only component of

$(C \setminus \{c_1, \ldots, c_k\}) \cup \{c_i\}$. Consequently, $D \setminus w_i = G \setminus (X_i \cup V(M_i))$ for every $i \in \{1, \ldots, l\}$ and hence there is a single positive integer n such that all graphs D_i are isomorphic with K_{2n} or $K_{n,n}$. It is not difficult to see that the last observation implies that the whole D is isomorphic with K_{2n+1} or $K_{n,n+1}$, respectively.

Our next aim is to show that $D = K_{n,n+1}$ is not possible, to this end we proceed by contradiction. Suppose that D is isomorphic with $K_{n,n+1}$. It is not difficult to see that since D_i is isomorphic with $K_{n,n}$ for every $i \in \{1, \ldots, l\}$, all vertices w_i, $i \in \{1, \ldots, l\}$, belong to the larger partite set of D. Recall that $l \geq 3$, let $M' = (M \setminus \{e_1, e_2, e_3\}) \cup \{f_1, f_2, f_3\}$, and let X' be an odd component of $(C \setminus \{u_1, \ldots, u_k\}) \cup \{u_1, u_2, u_3\}$. Clearly, G, M', and X' satisfy assumptions of Lemma 3.1.3 which in turn implies that $G \setminus (X' \cup V(M'))$ is isomorphic with K_{2m} or $K_{m,m}$ for some m. On the other hand, $G \setminus (X' \cup V(M')) = D - \{w_1, w_2, w_3\}$, and, since w_1, w_2, and w_3 all lie in the larger partite set of D, we have $D - \{w_1, w_2, w_3\} = K_{n,n-2}$, which is a contradiction. Therefore, we conclude that D is isomorphic with a complete graph.

Our final goal is to show that C is complete. Denote by C' the subgraph of G induced by the set of vertices $C \setminus \{u_1, \ldots, u_k\}$. Since D is an odd component of $G \setminus V(M)$,

the subgraph of G induced by C' is by Lemma 3.1.3 isomorphic with $K_{p,p}$ or K_{2p} for some integer p. Let y_1 and y_2 be arbitrary two vertices from $\{u_1, \ldots, u_k\}$ and x_1 and x_2 the two vertices of S joined by M to y_1, respectively to y_2. By Lemma 3.1.4 there are two independent edges, say x_1z_1 and x_2z_2, between $\{x_1, x_2\}$ and D. Let $M' = \{(M \setminus \{x_1y_1, x_2y_2\}) \cup \{x_1z_1, x_2z_2\}$. By Lemma 3.1.3 the graph $G \setminus (V(M') \cup D)$ is isomorphic with K_{2p+2} or $K_{p+1,p+1}$ for some integer p. Note that $G \setminus (V(M') \cup D) = C' \cup \{y_1, y_2\}$.

First we consider the case when C' contains a triangle, or, equivalently, C' is isomorphic with K_{2p} for some integer $p > 1$. Since C' contains a triangle and $C' \cup \{y_1, y_2\}$ is isomorphic with K_{2n} or $K_{n,n}$, we get that $C' \cup \{y_1, y_2\}$ is a complete graph irrespective of the choice of y_1 and y_2. It follows that C itself is a complete graph, in which case the proof is complete.

In the rest of the proof we show that the case that C' is isomorphic with a complete bipartite graph is not possible. We proceed by contradiction and suppose that that C' is isomorphic with $K_{p,p}$ for some integer p. Recall that $C' \cup \{y_1, y_2\}$ is either a complete, or a complete bipartite graph for any y_1 and y_2. First assume that $C' \cup \{y_1, y_2\}$ is a complete graph for some y_1 and y_2. Choosing $y_2 = x$ for all x in $C \setminus (C' \cup \{y_1\})$ shows that $C' \cup \{y'_1, y'_2\}$ is a complete graph for all pairs of y'_1 and y'_2 and hence C itself is a complete graph. In the rest of the proof we assume that $C' \cup \{y_1, y_2\}$ is isomorphic with $K_{p+1,p+1}$ for all pairs of y_1 and y_2. Comparing C' and $C' \cup \{y_1, y_2\}$ we observe that y_1 and y_2 lie in different partite sets of $K_{p+1,p+1}$. Since $k \geq 3$, there are at least three vertices in $C \setminus C'$, say v_1, v_2, and v_3. Using all the pairs from $\{v_1, v_2, v_3\}$ in turn in the roles of y_1 and y_2 we get that each two of $\{v_1, v_2, v_3\}$ lie in different partite sets of $K_{p+1,p+1}$, which is clearly not possible. This completes the proof. □

We conclude this section by showing that the requirement $|V(G)| \geq 2k+3$ in Lemma 3.1.6 cannot be relaxed. More precisely, for every $k \geq 3$ we construct a k-connected equimatchable factor-critical graphs with $2k+1$ vertices and a k-cut S such that $G - S$ has k components.

Proposition 3.1.10. *Let G be a k-connected equimatchable factor-critical graph with a k-cut S for some $k \geq 3$. Then $G - S$ has at most k components and this bound is tight for every $k \geq 3$.*

Proof. If $|V(G)| \geq 2k + 3$, then $G - S$ has exactly $2 \leq k$ components by Lemma 3.1.6. Therefore, we can assume that $|V(G) \leq 2k + 1$. Clearly, the number of components of $G - S$ is at most $|V(G - S)| \leq k + 1$, with equality if and only if $G - S$ consists from $k + 1$ singletons. However, it is easy to see that if $G - S$ consists from $k + 1$ singletons, then for arbitrary vertex s of S the graph $G - s$ cannot have a perfect matching, contradicting the factor-criticality of G.

To show that the bound is tight, for all $k \geq 3$ we construct a k-connected equimatchable factor-critical graph G_k with $2k+1$ vertices and a k-cut S such that $G - S$ has exactly k components. For arbitrary $k \geq 3$, consider a graph G_k with an independent set S of size k such that $G_k - S$ consists from $k - 1$ singletons and one copy K_2, which we denote by C. Finally, assume that every vertex of S is adjacent with every vertex of $G_k - S$ and that $G_k - S$ does not contain any edges except the edge in C. Clearly, the graph G_k is k-connected for every $k \geq 3$. To show that $G'_k = G_k \setminus \{v\}$ has a perfect matching for every vertex v, we distinguish whether v belongs to S, C, or is a singleton in $G_k - S$. If v is a vertex of S, then a perfect matching of G'_k can be constructed by taking the edge c_1c_2 from C and joining every vertex of $S \setminus \{v\}$ with a vertex in

$G'_k \setminus \{c_1, c_2\}$. If v is a vertex of C or a singleton of $G_k - S$, then it is clearly possible to joint every vertex in S with a vertex in $G'_k - S$ to obtain the desired perfect matching.

To show that any matching M of G_k can be extended to a maximum matching, we distinguish two cases according to whether M contains the edge of C or not. If M contains the edge c_1c_2 of C, then $G_k - \{c_1, c_2\}$ is isomorphic with $K_{k,k-1}$ and hence any matching containing c_1c_2 can be extended to an almost-perfect matching of G_k. If M does not contain any edge from C, then M contains only edges from $G_k - E(C)$, which is isomorphic to $K_{k,k+1}$ and again M can be extended to an almost-perfect matching of G_k, which completes the proof. □

3.2 Graphs with independence number 2

In this section we investigate the relationship between equimatchability and independence number. We focus on odd k-connected graphs with $k \geq 4$, at least $2k+3$ vertices, and a k-cut which separates at least two components with at least 3 vertices and, perhaps surprisingly, show that such graphs are equimatchable factor-critical if and only if their independence number equals 2. In one direction, we show that if a graph with independence number 2 is odd, then it is equimatchable, and if it is even, then it is very close to being equimatchable. In the reverse direction, we use the characterisation of k-connected equimatchable factor-critical graphs with at least $2k + 3$ vertices and a k-cut separating at least two components with at least three vertices from Theorem 3.1.9 to show that if $k \geq 4$, then all such graphs have independence number 2. Finally, we provide examples showing that it is not possible to extend these results to graphs in which every minimal cut separates a component with at most 2 vertices – even if such graphs are equimatchable factor-critical, they can have arbitrarily large independence number. Note that Proposition 3.1.10 from the previous section shows that these result can neither be extended to graphs with at most $2k + 1$ vertices, since in such graphs $G - S$ can have k components and hence also independence number at least k.

Lemma 3.2.1. *Let G be a k-connected factor-critical equimatchable graph with a k-vertex cut S such that $G - S$ has precisely two components C and D, both of them complete. Furthermore, let $\{c, d, s\}$ be vertices of G such that $c \in C$, $d \in D$, and $s \in S$. If there is a matching M such that M covers all vertices of $S - s$, both $C - V(M)$ and $D - V(M)$ are odd, and $V(M) \cap \{c, d, s\} = \emptyset$, then the subgraph of G induced by $\{c, d, s\}$ contains at least one edge.*

Proof. Since both C and D are complete and both $C - V(M)$ and $D - V(M)$ are odd, the subgraphs of G induced by $C - V(M)$ and $D - V(M)$ are odd complete graphs. Therefore, there are matchings M_C and M_D of $C - V(M)$ and $D - V(M)$ covering all vertices of $C - (V(M) \cup \{c\})$, respectively $D - (V(M) \cup \{d\})$. It follows that $M' = M \cup M_C \cup M_D$ is a matching of G covering all vertices of G except c, d, and s. Since G is factor-critical and equimatchable, M' can be extended to a maximum matching of G, that is, a matching covering all but one vertices of G. Consequently, the subgraph of G induced by $\{c, d, s\}$ contains at least one edge, as claimed. □

Lemma 3.2.2. *Let G be a k-connected factor-critical equimatchable graph with a k-cut S for some $k \geq 3$. Assume that $G - S$ has precisely two components C and D, both of them complete. Let s be a vertex of S. Then there exists a matching M such that M contains only edges of the subgraph of G induced by $S - s$ and $|S - V(M)| = 3$ if k is odd and $|S - V(M)| = 2$ if k is even.*

Proof. Assume that k is even. One of the components of $G - S$, say C, is odd and the other is even. Let $e = sc$ be an edge between C and s, M_D a perfect matching of D, and M_C a perfect matching of $C - c$. Consider the set $N = M_D \cup M_C \cup \{e\}$. Since N is a matching and G is equimatchable and factor-critical, N can be extended to a matching N' such that N' leaves only one vertex of S uncovered. Clearly, $M = N' - N$ is a matching that satisfy the assumption of the lemma.

Now assume that k is odd. It is easy to see that both C and D have the same parity. If both are even and M_C and M_D are perfect matchings of C and D, respectively, then the matching $N = M_C \cup M_D$ can be extended to a matching N' such that N' leaves only one vertex $s' \in S$ uncovered. Denote $N' - N$ by M. If $s = s'$, then let e be an arbitrary edge of M. Otherwise, let e be a edge in M incident with s. Then $M - e$ is the desired matching.

Finally, assume that both components of $G-S$ are odd. Let $s' \in S$ be a vertex different from s and let $e = sc$ and $f = s'd$ be arbitrary edges such that $c \in C$ and $d \in D$. Moreover, let M_C and M_D be any perfect matchings of $C - c$ and $D - d$, respectively. Then the matching $N = M_C \cup M_D \cup \{e, f\}$ can be extended to a matching N' such that N' leaves only one vertex of S uncovered. It is easy to verify that $N' - N$ is the desired matching, which completes the proof. □

Lemma 3.2.3. *Let G be a k-connected factor-critical equimatchable graph with a k-cut S for some $k \geq 4$. Assume that $G \setminus S$ has two components C and D with at least 3 vertices. Then for any vertices $s \in S$, $c \in C$, and $d \in D$ there is a matching of G covering all vertices of $G \setminus \{c, d, s\}$.*

Proof. Theorem 3.1.9 implies that both C and D are complete and that $G - S$ does not have any other components.

We will need the following observation:

Claim. If s' is adjacent to only one vertex of some component of $G - S$, then s' is adjacent to all vertices of the other component of $G - S$.

Proof of Claim. For a contradiction suppose that d is the only vertex of D adjacent to s'. Let $R = (S \cup \{d\}) - \{s'\}$ and note that R is a k-cut of G such that $G - R$ has two components, namely $C \cup \{s'\}$ and $D - \{d\}$. If $|D| = 3$ by our assumptions the components of $G - R$ have sizes 2 and at least $k + 1$. By Theorem 3.1.8 either $R \cup C \cup \{s'\}$ is a complete bipartite graph minus a matching or $C \cup \{s'\}$ is a complete graph. Since C is a complete graph, $R \cup C \cup \{s'\}$ cannot be bipartite and the claim follows. For the rest of the proof assume that $|D| \geq 4$. Since by our assumptions G has at least $2k + 3$ vertices and the components of $G - R$ have both at least 3 vertices, by Theorem 3.1.9 both these components are complete. In particular, s' is adjacent to all vertices of C. The proof of the claim is now complete.

By Lemma 3.2.2 there is a matching M covering vertices of $S - s$ such that $|S - V(M)| \leq 3$. Let $C' = C - \{c\}$, $D' = D - \{d\}$, and $S' = S - V(M) - \{s\}$. We distinguish three cases depending on parity of C and D.

First, let k be odd and both components of $G - S$ even. Since k is odd, by Lemma 3.2.2 has S', denote them by s_1 and s_2 If both s_1 and s_2 have a neighbour in D', then by Lemma 3.1.4 there are two independent edges between S' and C. Let e be an edge between S' and C which does not contain vertex c. Without a loss of generality we can assume that the vertex of S' covered by e is s_1. Let f be an edge between s_2 and D'. If there exists an edge f between s_2 and D', then the matching $M \cup \{e, f\}$ satisfy the assumption of Lemma 3.2.1 and we are done. Therefore, we can assume that s_2 does not have a neighbour in D'. Then by Lemma 3.1.4 there is an edge g between s_1 and D' and by Claim there is and edge h between s_2 and C'. The proof of this case is completed by applying Lemma 3.2.1 to the matching $M \cup \{g, h\}$.

If both components of $G - S$ are odd, then our aim is to to match both vertices of S' with the same component. If we can do it, then we are done by Lemma 3.2.1. If it is not possible, then either one vertex of S', say s_1, is not adjacent to any vertex of C', or both vertices, or both vertices of S' have two neighbours in C. In the first case s_1 is adjacent to every vertex of D and hence s_2 have to be adjacent only with d and therefore s_2 is adjacent to every vertex of C. In the second case is easy to see that both s_1 and s_2 must have same two neighbours in C and also in D. In both cases, we can always choose matchings N_C and N_D between S' and C and between S' and D, respectively, in such way that s_1 is matched with c by N_C and s_2 is matched with d by N_D, or we can switch the labels for vertices s_1 and s_2. Since $k \geq 4$, S contains an edge s_3s_4 in M. By Lemma 3.1.4 there is a matching M_C between $\{s_1, s_2, s_3\}$ and C and a matching M_D between $\{s_1, s_2, s_4\}$ and D. It is easy to verify that M_C and M_D can be chosen in such way that M_C matches s_1 with c and M_D matches s_2 with d. The matching $(M - \{s_3s_4\}) \cup (M_C - \{s_1c\}) \cup (M_D - \{s_2d\})$ satisfy the assumptions of Lemma 3.2.1, whose application completes the proof of this case.

Finally, let k be even. By Lemma 3.2.2 S' has only one vertex, denote it by s_1. Since k is odd, one of the components of $G - S$ has to be even and the other odd. We can assume that C is even. If s_1 is adjacent to some vertex in C', then we are done by Lemma 3.2.1. Otherwise, c is the only neighbour of s in C and by Claim the vertex s_1 is adjacent to all vertices of D. Since $k \geq 4$, there is an edge s_2s_3 in M. Vertices s_2, s_3 are both from S. By Lemma 3.1.4 there are two independent edges between $\{s_2, s_3\}$ and D. At least one of the edges, say the one incident with s_2, does not have d as an endvertex. Denote this edge by e. By the same lemma there are two independent edges between $\{s_1, s_3\}$ and C. Clearly, one of these edges is s_1c. Hence, there is an edge f between s_3 and C'. Since $|D| \geq 3$ and s_1 is adjacent to all vertices of D, there is an edge g between D' and s_1 which is not adjacent to e. Applying Lemma 3.2.1 to the matching $(M - \{s_2s_3\}) \cup \{e, f, g\}$ finishes the proof. $\square$

Lemma 3.2.4. *Let G be a k-connected factor-critical equimatchable graph for some $k \geq 4$. Assume that G has at least $2k+3$ vertices and a k-cut such that $G \setminus S$ has two components C and D with at least 3 vertices. Then every vertex $s \in S$ is adjacent to either every vertex of C or every vertex in D.*

Proof. We prove the lemma by contradiction. Suppose that there is a vertex $s \in S$ that is adjacent neither to all vertices of C, nor to all vertices of D. It follows that there are two vertices $c \in C$ and $d \in D$ such that s is adjacent neither to c nor to d. By Lemma 3.2.3 there is a matching M of G covering all vertices of $G - \{c, d, s\}$. Since S is a cut, c and d are not adjacent. Hence M is a maximal matching of G leaving unmatched 3 vertices, contradicting the fact that G is factor-critical and equimatchable. □

Lemma 3.2.5. *Let G be a k-connected factor-critical equimatchable graph with a k-cut S for some $k \geq 4$ such that $G - S$ has two components with at least 3 vertices. Then the independence number of G is 2.*

Proof. By Theorem 3.1.9 both C and D are complete and $G - S$ does not have any other components. Since both C and D are complete, no independent set of G can contain more than one vertex from each of them. We distinguish three cases according to the type of a possible independent set I of size 3 in G.

Case a) $I = \{c, d, s\}$, where $c \in C$, $d \in D$, and $s \in S$. This is not possible by Lemma 3.2.3.

Case b) $I = \{s_1, s_2, s_3\}$, where all vertices of I belong to S. If C is odd, let e be an edge joining a vertex $s \in S - I$ to C, such an edge exists by Lemma 3.1.4. If D is odd, let f be an edge joining $s' \in S - (I \cup V(e))$ to D. If D is odd, the edge f always exists since then also S is odd, there is a vertex s' in $S - (I \cup V(e))$, and s' is joined by an edge to D by Lemma 3.1.4. Clearly, there is a perfect matching M_C of $C - V(e)$ and a perfect matching M_D of $D - V(f)$. It follows that $M = M_C \cup M_D \cup \{e, f\}$ is a matching of G covering all vertices except s_1, s_2 and s_3. Since G is factor-critical and equimatchable, M can be extended to a matching of G leaving uncovered exactly one vertex, which implies that there is an edge in I and thus I cannot be an independent set.

Case c) $I = \{s_1, s_2, x\}$, where $\{s_1, s_2\} \subseteq S$ and x is a vertex of either C or D. This case is very similar to Case b) and thus ommited.

It follows that I cannot be of any of the three possible types, implying that the maximum size of an independent set of G is at most 2. Clearly, G cannot be a complete graph and hence it has an independent set with size precisely two, which completes the proof. □

The following two propositions show a close relationship between equimatchable and almost-equimatchable graphs, and graphs with independence number 2.

Proposition 3.2.6. *Let G be a graph with independence number 2. If G is odd, then G is equimatchable. If G is even, then G is either K_{2n} or $K_{n,n}$ for some nonnegative integer n, or is not equimatchable, has a perfect matching, and every matching of G misses at most 2 vertices.*

Proof. For any maximal matching M, the set of vertices not covered by M induces an independent set. Hence any maximal matching of a graph with independence number 2 leaves uncovered at most 2 vertices. Since the parity of the number of vertices not covered by a matching is the same as the parity of the number of vertices of the graph, if G is odd, then any maximal matching of G leaves uncovered exactly one vertex. Consequently, all maximal matchings of G have the same size and G is equimatchable. If G is even, then every maximal matching of G leaves uncovered 0 or 2 vertices. We distinguish two cases: either G is equimatchable, or not. If G is equimatchable with a perfect matching, then it is isomorphic with K_{2n} or $K_{n,n}$ for some nonnegative integer n by [39]. Suppose that G is equimatchable and every maximal matching of G leaves uncovered exactly 2 vertices, our aim is to show that there are no such graphs. Since G is odd, it cannot be factor-critical. Furthermore, G does not have a perfect matching and thus it has a nontrivial Gallai-Edmonds decomposition. It is well known that the number of vertices uncovered by any maximum matching equals the difference of the number of components in C and the number of vertices in A, see for example [22]. It follows that there are at least components in C, which contradicts the fact that the independence number of G is 2. The only remaining possibility is that G is not equimatchable, in which case G has a perfect matching and a maximal matching leaving uncovered precisely, and every maximal matching of G leaves uncovered at most 2 vertices. □

Odd graphs with independence number 2 are described in the following proposition.

Proposition 3.2.7. *Let G be an odd graph with independence number 2. Then G is either factor-critical, or an union of two complete graphs, one even and one odd, joined by a set of pairwise incident edges.*

Proof. If G is 2-connected, then by Gallai-Edmonds decomposition G is either bipartite, or factor-critical. If G is factor-critical, then there is nothing to prove. Therefore, we can assume that G is bipartite. Since each partite set of a bipartite graph is independent, both partite sets of G have size at most 2. From the fact that G is odd follows that $G = P_3$, and hence G is an union of K_1 and K_2 joined by an edge, as claimed.

If G has a cutvertex v, then $G - v$ has exactly two components, otherwise the independence number of G would be at least 3. Moreover, both components of $G - v$ are complete, since otherwise there would be an independence set with size 3 consisting from two nonadjacent vertices of one component and any vertex of the second component. If there are two vertices u and w from different components of $G - v$ such that v is not adjacent with neither of them, then again $\{u, v, w\}$ is an independent set of size 3. Hence v is adjacent with every vertex of at least one component of $G - v$ and G is an union of two complete graphs, one even and one odd, joined by a set of pairwise incident edges. □

Theorem 3.2.8. *Let G be a k-connected graph for some $k \geq 4$ with an odd number of vertices and a k-cut S such that $G - S$ has two components with at least 3 vertices. Then G is factor-critical and equimatchable if and only if its independence number is 2.*

Proof. If G is equimatchable and factor-critical, then its independence number is 2 by Lemma 3.2.5.
In the reverse direction, assume that G has independence number 2. Then G is equimatchable by Proposition 3.2.6. The characterisation of equimatchable graphs [19] asserts that any 2-connected equimatchable graph is either factor-critical, or bipartite. If G is factor-critical, then we are done. If G is bipartite, than each partite set forms an independent set and hence the size of each partite set is at most 2. Since G is odd, it follows that G has at most 3 vertices and hence it cannot be k-connected for any $k \geq 4$. □

Our final result shows that Lemma 3.2.5 cannot be extended to equimatchable graphs without two components with at least 3 vertices.

Theorem 3.2.9. *For every triple of integers n, k, and m such that $k \geq 3$ and $m \in \{1, 2\}$ there is a k-connected equimatchable factor-critical graph G with a k-cut S such that $G - S$ has a component of size m and G has an independent set of size at least n.*

Proof. First assume that $m = 1$. Let $l = \max\{n, k\}$ and denote by H a copy of $K_{l,l}$. Choose a set S of k vertices of H in such a way that S contains at least one vertex from both partite sets of H. The desired graph G is constructed by taking a new vertex v and joining it with every vertex in S. and by H_2 a copy of K_2. Clearly, G is k-connected and S is a k-cut of G. Since v is a component of $G - S$ and $m = 1$, the graph $G - S$ has a component with m vertices. Furthermore, it is easy to directly verify that G is factor-critical and equimatchable. The proof of this case is concluded by observing that each partite set of H forms an indepentendent set of G with size $l \geq n$.

Now we assume that $m = 2$. Let $l = \max\{n, k\}$ and denote by H_1 a copy of $K_{l,l+1}$ and by H_2 a copy of K_2. Denote by X a set of k vertices from the larger partite set of H_1. The desired graph G is constructed by joining both vertices of H_2 with all vertices of X. It can be easily verified that the resulting graph is k-connected, equimatchable, and factor-critical. Clearly X is a k-cut of G such that $G - X$ has a component with m vertices. Finally, G contains an independent set with $l + 1 \geq n$ vertices, which completes the proof. □

4

Stable, critical, and extremal equimatchable graphs.

This chapter is devoted to extremal properties of graphs related to equimatchability. First we characterize all minimum and maximum graphs with respect to edge removal, respectively addition. Second we investigate critical equimatchable graphs, that is, graphs which are equimatchable, but removing any vertex yields a graph which is not equimatchable. Finally, we investigate graphs G such that $G - v$ is equimatchable for every vertex v of G. These graphs fall naturally into one of two disjoint classes – stable equimatchable graphs and vertex-minimal non-equimatchable graphs.

A graph G is *edge-minimal non-equimatchable* if G is not equimatchable and $G - e$ is equimatchable for every edge e of G. Similarly, G is *edge-maximal non-equimatchable* if G is not equimatchable and $G \cup e$ is equimatchable for every edge e of G. We start by determining all edge-minimal and edge-maximal non-equimatchable graphs.

Theorem 4.0.1. *The only edge-minimal non-equimatchable graph is* P_4.

Proof. Our proof starts with the observation that if for every edge $e \in E(G)$ the graph $G - \{e\}$ is equimatchable and G has a vertex v of degree at least three, then G is equimatchable. To prove this, consider a matching M of G. Clearly, at least two edges incident with v are not in M. Moreover, there is a maximum matching N of G that does not contain at least one of these two edges, denote it by f. Clearly, N is a matching of $G - \{f\}$ and it is easy to see that N has to be maximum matching of $G - \{f\}$. Hence, the maximum size matchings of G and $G - \{f\}$ have the same size. Since $G - \{f\}$ is equimatchable, the matching M can be extended to a maximum matching of $G-\{f\}$, which is also a maximum matching of G. Therefore, every vertex of G has degree at most two and G is a cycle or a path. Hence, if e is an edge of G, then $G-\{e\}$ is a path or two disjoint paths. It is easy to see that a path is equimatchable if and only if it has at most three vertices. It follows that, first, G is not a path with at most 3 vertices, and second, that removing any edge from G yields one or two paths with at most 3 vertices. Since C_3 is also equimatchable, the only suitable graph is P_4. □

The join of two disjoint graphs G and H is the union $G \cup H$ together with all the edges that connect $V(G)$ and $V(H)$.

Theorem 4.0.2. *If G is an edge-maximal non-equimatchable graph, then G is isomorphic with K_{2n} minus an edge or K_{2n+1} minus a triangle, where n is some positive integer.*

Proof. Let N be a maximal matching of G which is not maximum. Since G is not equimatchable, such a matching exists. Let u and v be two vertices of N which are not adjacent. Clearly, N is maximal matching of $G \cup \{uv\}$ which is not maximum. Since by the assumption of the theorem $G \cup e$ is equimatchable for every $e \notin E(G)$, it follows that such a pair of nonadjacent vertices u, v cannot exist. Therefore, the vertices of N form a copy of K_{2n} for some integer $n \geq 1$. Since N is maximal, the graph $G - V(N)$ does not contain an edge and because N is not maximum matching, the graph $G - V(N)$ contains at least two vertices. Denote $|G - V(N)|$ by k. Let n be a vertex of $V(N)$ and x a vertex of $G - V(N)$.

Our next aim is to show that for arbitrary choice of vertices n and x the graph G contains the edge nx. For contradiction suppose that nx is not an edge of G. Clearly, N is a maximal matching of $G \cup \{nx\}$, that is not maximum. Consequently, if f is not an edge of G, then f has both endvertices in $G - V(N)$. The graph $G_f = G \cup \{f\}$ consists of a copy of K_{2n+2} and a set of $k-2$ independent vertices. It is easy to verify that if $k-2 \leq 1$, then G_f is equimatchable. We finish the proof by showing that if $k-2 \geq 2$ and G is a join of K_{2n} with k isolated vertices, then G_f is not equimatchable. Since every of $k-2$ independent vertices is adjacent to every vertex of N, then there exist a matching M between at least two of independent vertices and vertices of N. Clearly, the matching M can be extended to a matching with at least

$n+2$ edges. However, $N \cup \{f\}$ is a maximal matching with $n+1$ edges and G_f is not equimatchable. The proof is now complete. □

An equimatchable graph is called *critical* if $G-v$ is not equimatchable for every vertex v of G. The following theorem provides a necessary condition for an equimatchable graph to be critical.

Theorem 4.0.3. *If G is a critical equimatchable graph, then G is factor-critical and does not contain a vertex v such that $G-v$ is isomorphic with K_{2n} or $K_{n,n}$.*

Proof. In the proof we distinguish three cases according to type of Gallai-Edmonds decomposition of G. Recall that Gallain-Edmonds decomposition of a graph G is trivial, that is, all vertices of G belong to a single set of the decomposition, if and only if either G has a perfect matching, or G is factor-critical. First assume that G has a perfect matching. By Theorem 1.2.2 G is complete or complete regular bipartite graph. It is easy to verify that after for arbitrary vertex v of G is the graph $G-v$ complete or complete bipartite and hence equimatchable graph. Therefore, any equimatchable graph with a perfect matching cannot be critical equimatchable.

Second assume that G is factor-critical. Then $G-v$ has a perfect matching, for arbitrary vertex $v \in V(G)$. If it is not critical equimatchable, then there exists a vertex $v \in V(G)$ such that $G-v$ is a randomly matchable graph, hence by Theorem 1.2.2 isomorphic with K_{2n} or $K_{n,n}$. Therefore, every equimatchable factor-critical graph G, which does not contain a vertex v such that $G-v$ is isomorphic with K_{2n} or $K_{n,n}$, is a critical equimatchable graph.

Finally assume that G has non-trivial Gallai-Edmonds decomposition. To finish the proof we show that every such G has a vertex v such that $G-v$ is equimatchable. Let (D, A, C) be the Gallai-Edmonds decomposition of G. We show that for arbitrary vertex $a \in A$ is the graph $G-a$ equimatchable. It is easy to see that Gallai-Edmonds decomposition of $G-a$ is $(D, A-a, C)$. It is sufficient to verify the assumptions of Theorem 1.2.6. Since G is equimatchable, then C is empty, $A-a$ is an independent set, all components of D are of correct type and there are more singletons and type I components in D that vertices in $A-a$. It remains to proof that the bipartite graph obtained from $G-a$ by contracting all components of D to singletons and deleting all points corresponding to type II and III components of D is equimatchable. This is equivalent to show that if we remove a vertex from smaller parite set of equimatchable bipartite graph, then we get equimatchable graph.

Let $G' = (U, W)$ be a connected bipartite equimatchable graph with $|U| \leq |W|$ and x a vertex of U. Due to Theorem 1.2.7, for every vertex in $u \in U$, there exists a non-empty $X \in N(u)$ such that $|N(X)| \leq |X|$ and hence $|N(X) - x| \leq |X|$. Therefore, by Theorem 1.2.7 the graph $G' - x$ is equimatchable. Hence, every graph G with non-trivial Gallai-Edmonds decomposition is not critical equimatchable, which completes the proof. □

The rest of the chapter is devoted to graphs in which removing arbitrary vertex yields an equimatchable graph. Our results allow us to completely determine stable equimatchable graphs – graphs which are equimatchable and removing arbitrary vertex yields an equimatchable graph. On the other hand, if such a graph is not equimatchable, that is, it is vertex-minimal non-equimatchable, then it necessarily has a perfect matching. However, characterisation of such graphs with a perfect matching seems to be out of reach of the existing methods.

Theorem 4.0.4. *Let G be a connected factor-critical graph. The graph $G - v$ is equimatchable for every vertex $v \in V(G)$ if and only if G is isomorphic with K_{2n+1} for some n.*

Proof. Clearly, after removing of an arbitrary vertex of K_{2n+1} we get a complete graph and complete graphs are equimatchable.

Conversely, assume that the graph $G - v$ is equimatchable for every vertex $v \in V(G)$. We show that arbitrary two vertices x, y of G are adjacent. First note that if we remove a vertex x from a factor-critical graph we get a graph with an perfect matching. Therefore, removing arbitrary vertex from G yields an equimatchable graph with a perfect matching.

If we remove a vertex x from factor-critical graph G we get a graph with an perfect matching. Let M be a perfect matching of $G - \{x\}$ and z the vertex paired with y in M. Clearly, $M' = M - \{yz\}$ is a matching of $G - \{z\}$. Hence M' can be completed to a perfect matching and xy are adjacent. This completes the proof. □

Theorem 4.0.5. *Let G be a connected graph and (D, A, C) be its Gallai-Edmonds decomposition and suppose $A \neq \emptyset$. If for every vertex $v \in V(G)$ the graph $G - v$ is equimatchable, then G is bipartite graph.*

Proof. We begin by proving that every component of D is a singleton. For contradiction suppose that there is a component D_i of the subgraph of G induced by D such that D_i has at least three vertices. If we fix a particular component D_i of D, with $|D_i| \geq 3$. Let $d \in D_i$ be a vertex adjacent to some vertex $a \in A$, then there exists a maximum matching M of G not covering the vertex d. By Theorem 1.2.3 if M is a maximum matching of G, then it contains a near-perfect matching of each component of D, a 1-factor of each component of C, and matches all vertices of A with vertices in distinct components of D.

Let v be a vertex of D_i different from d, M_v a perfect matching of $D_i - v$, s the vertex of D_i matched with d by M_v, and t the vertex matched with a by M. Let M' be a matching obtained from M by replacing the near-perfect matching of D_i by M_v. The matching M' is a maximum matching of G and hence a maximum matching of $G - v$. Consider the set $M'' = (M' - \{at, ds\}) \cup \{ad\}$. It is easy to verify that matching M'' is a maximal matching of $G - v$, but it is not maximum. Therefore, the graph $G - v$ is not equimatchable.

Our next goal is to show that if every component of D is a singleton, then G is bipartite. Let $\{d\}$ be a component of D. We distinguish two cases according to whether $G - d$ has a perfect matching, or not.

First assume that $G - d$ does not have a perfect matching. It is easy to verify that the Gallai-Edmonds decomposition of the graph $G - d$ is $(D - d, A, C)$. Since $G - d$ is equimatchable, C is empty and A is independent set. Therefore, G is bipartite with parite sets A and D.

Second assume that $G - d$ has a perfect matching. Since $G - d$ is equimatchable, every its component is even complete or regular complete bipartite graph. Now we show that $G - d$ cannot have a complete component. Suppose for the contrary that X is a component of $G - d$ that is complete and $x \in X$ a vertex adjacent to d. Since K_2 is also a regular complete bipartite graph, we can assume that $|X| \geq 3$. We show that every vertex of $X - x$ is not covered by some maximum matching of G and hence $X - x$ is a component of D and thus a singleton, which contradicts the fact that

$|X| \geq 3$. Let $y \in X$ be a vertex different from x and M_y be a perfect matching of all components of $G - d$ such that x is matched with y. It is easy to see that the matching $(M_y - \{xy\}) \cup \{xd\}$ is a maximum matching of G that leaves y uncovered.

It remains to prove that d is adjacent to only one partition of every component of $G - d$. On the contrary, suppose that d is adjacent to edge xy of $G - d$ and let M be a perfect matching of $G - d$ that contains edge xy. Since $G - d$ is an equimatchable graph with a perfect matching, such matching M always exists. Clearly, M is a maximum matching of G. Moreover, $(M - \{xy\}) \cup \{xd\}$ and $(M - \{xy\}) \cup \{yd\}$ are maximum matchings of G that leave uncovered y and x, respectively. Hence, vertices x, y, and d are in the same component of D, which contradicts the fact that every component of D is singleton. This finishes the proof. □

Theorem 4.0.6. *Let $G = (U, W)$ be a connected bipartite graph with $|U| < |W|$. The graph $G - v$ is equimatchable for every $v \in V(G)$ if and only if, for all $u \in U$, there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| < |X|$.*

Proof. First assume that for all $u \in U$, there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| < |X|$. Let x and y be an arbitrary vertices of U and W, respectively. For every vertex u of U, there exists a non-empty $X subseteq N(u)$ such that $|N(X)| < |X|$ and hence $|N(X) - x| < |X|$. Furthermore, $|N(X - y)| \leq X - y$ regardless of whether $y \in X$ or not. Therefore, by Theorem 1.2.7 both $G - x$ and $G - y$ are equimatchable, which completes the proof of one implication.

In the reverse direction, assume that $G - v$ is equimatchable for every vertex $v \in V(G)$. If u is a vertex of U adjacent with all vertices of W, then let $X = W$. Clearly, $X \subseteq N(w)$ and $N(X) = U$, hence $|N(X)| < |X|$ and we are done.

Assume that u is a vertex of U which is not adjacent to all vertices of W. Let w be a vertex of W not adjacent to u. Then $G - w$ is a bipartite equimatchable graph and by Theorem 1.2.7 there exists a non-empty $X \in N(u)$ such that $|N(X)| \leq |X|$. We proceed by contradiction. Suppose that there does not exists a non-empty set $X \in N(u)$ such that $|N(X)| < |X|$. Consider the set $S = \{X | X \in N(u) \text{ and } |N(X)| = |X|\}$.

Our next aim is to show that there is a vertex x from $N(u)$ contained in all sets of S. Let A, B be two distinct sets of S. Since u is an element of $N(X)$ for every $X \in S$, it is easy to verify that if A and B are disjoint, then $A \cup B$ is a set of neighbours of u

satisfying $|N(A \cup B)| < |A \cup B|$. Therefore, the intersection of any two sets in S is not empty. Since $A \cup B$ is set of neighbours of u, we can assume that $|N(A \cup B)| \geq |A \cup B|$. Then we have

$$|N(A \cup B)| = |N(A)| + |N(B)| - |N(A) \cap N(B)| =$$
$$|A| + |B| - |N(A) \cap N(B)| \geq |A \cup B| = |A| + |B| - |A \cap B|.$$

Hence, $|A \cap B| \geq |N(A) \cap N(B)|$ and it is easy to verify that $N(A) \cap N(B) \supseteq N(A \cap B)$. Therefore, if it holds that $|N(A \cup B)| \geq |A \cup B|$, then $A \cap B$ is a set of neighbours of u such that $|N(A \cap B)| \leq |A \cap B|$. Since there does not exists a non-empty set $X \in N(u)$ such that $|N(X)| < |X|$, the set $A \cap B$ belongs to S. It is easy to show by an induction that $\cap_{X \in S} X$ is a non-empty set from S and thus there exists a vertex $x \in N(u)$ such that for every $X \in S$ is $x \in X$. Since for every non-empty set $X \in N(u)$ holds $|N(X)| \geq |X|$ and every set in S contains x, in the graph $G - x$ there does not exists a non-empty set $X \in N(u)$ such that $|N(X)| \leq |X|$. Therefore, by Theorem 1.2.7 $G - x$ is not equimatchable, which completes the proof.

□

Recall that an equimatchable graph G is *stable* if $G - v$ is equimatchable for every vertex $v \in G$. Using Theorem 4.0.4, 4.0.5, and 4.0.6 we are now able to characterise all stable equimatchable graphs.

Theorem 4.0.7. *Let G be a connected stable equimatchable graph. Then G is either K_n or $K_{n,n}$ for some positive integer n, or G is a bipartite graph (U, W) with $|U| < |W|$ such that for all $u \in U$, there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| < |X|$.*

Proof. In the proof we distinguish three cases according to type of Gallai-Edmonds decomposition of G.

First assume that G is a stable factor-critical graph. Then by Theorem 4.0.4 G is isomorphic with K_{2m+1} for some positive integer m.

Second, let G be a stable graph with a perfect matching. Since G is equimatchable, by Theorem 1.2.2 G is isomorphic with K_{2n} or $K_{n,n}$ for some positive integer n. It is easy to verify that in both cases the graph G is stable.

Finally, if G has non-trivial Gallai-Edmonds decomposition, then by Theorem 4.0.5

G is bipartite. Assume that $G = (U, W)$. Recall that if G is equimatchable and $|U| = |W|$, then G has a perfect matching. For the rest of the proof assume that $|U| < |W|$. By Theorem 4.0.6 for all $u \in U$, there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| < |X|$. By Theorem 1.2.7 such bipartite graph is equimatchable, which finishes the proof. □

Regarding vertex-minimal non-equimatchable graphs, that is, graphs which are not equimatchable, but removing an arbitrary vertex yields an equimatchable graphs, Theorem 4.0.4, 4.0.5, and 4.0.6 allow us to prove that any such graph necessarily has a perfect matching.

Theorem 4.0.8. *If G is connected vertex-minimal non-equimatchable graph, then G has a perfect matching.*

Proof. By Theorem 4.0.4 if G is factor-critical vertex-minimal non-equimatchable graph, then G is equimatchable. Theorem 4.0.5 and Theorem 4.0.6 show that every vertex-minimal non-equimatchable graph without a perfect matching has to be bipartite with partite sets of the same size. It remains to show that all such graphs have a perfect matching.

Assume that $G = (U, W)$ is a bipartite vertex-minimal non-equimatchable graph with $|U| = |W|$. Let u be an arbitrary vertex of U. Since $G - u$ is equimatchable there is a matching of size $|U| - 1$ covering whole $U - u$ and the Gallai-Edmonds decomposition of $G - u$ is $(W, U - u, \emptyset)$. Therefore, for an arbitrary neighbour w of u there exists a maximum matching M of $G - u$ that leaves w uncovered. It is easy to see that $M \cup \{uw\}$ is a perfect matching of G, which completes the proof. □

As mentioned earlier, the following problem remains open.

Problem. *Characterize vertex-minimal non-equimatchable graphs with a perfect matching.*

Regarding the relationship of the preceding open problem with other results, let Δ_1 be the class of graphs which are not equimatchable and in which every matching can be extended to a matching with size at most one less than the size of a maximum matching. Note that with this notation, all even graphs with independence number 2 which are not isomorphic with K_{2n} or $K_{n,n}$ belong to Δ_1, see Proposition 3.2.6.

It is easy to see that every vertex-minimal non-equimatchable graph with a perfect matching belongs to Δ_1. On the other hand, not every graph with a perfect matching from Δ_1 is vertex-minimal non-equimatchable since C_6 is in Δ_1 and path on 5 vertices is not equimatchable.

The class Δ_1 was introduced and investigated in [18]. While [18] gives a characterisation of Δ_1 graphs without a perfect matching in terms of their Gallai-Edmonds decomposition, characterising Δ_1 graphs with a perfect matching seems to be a much more difficult problem, even for graphs with independence number 2. Therefore, determining vertex-minimal non-equimatchable graphs with a perfect matching may provide the first step towards understanding of the Δ_1 graphs with a perfect matching.

Conclusion

This thesis deals with extendability of matchings in graphs. Among the most prominent concepts of extendability is equimatchability - a graph is equimatchable if every its matching can be extended to a maximum matching. Equimatchable graphs fall naturally into two disjoint classes – factor-critical or bipartite and this thesis investigate mostly equimatchable factor-critical graphs.

Our main results are the following theorem and its three applications described below. While it is not difficult to prove that for any matching M isolating a vertex v of G all components of $G - (V(M) \cup v)$ are K_{2n} or $K_{n,n}$ the following theorem shows that the rest of the graph is connected.

Theorem. *Let G be a 2-connected, factor-critical equimatchable graph. Let $v \in V(G)$ be a vertex of G and M_v minimal matching that isolates v. Let $G' = G \setminus (V(M_v) \cup \{v\})$. Then G' is isomorphic with K_{2n} or a $K_{n,n}$ for some nonnegative integer n.*

The first application of the above theorem is determination of the maximum size of a 2-connected equimatchable factor-critical graph embeddable in the surface with genus g. We prove that for sufficiently large g the size is between $4\sqrt{2g}$ and $8\sqrt{g}$ for orientable surfaces and similar bounds are obtained also for nonorientable surfaces. These results improve the previously best result $O(g^{3/2})$ for 3-connected graphs. Note that the situation with 2-connected bipartite equimatchable graphs is different – while it was known that there are no 3-connected bipartite equimatchable graphs embeddable with face-width at least 3, we show that for every surface there are infinitely many 2-connected bipartite equimatchable graphs embeddable in the surface with arbitrarily large face-width.

The second application of the main theorem is showing that for every k there are only finitely many k-degenerate 2-connected equimatchable factor-critical graphs. Graphs with bounded degeneracy form a prominent class of somwhere-dense graphs, generalizing many important nowhere-dense graph classes, including graphs of bounded treewidth, graphs of bounded genus, graphs with a given set of forbidden minors or topological minors, and graphs with bounded expansion. In particular, our result generalizes the previous result that there are only finitely many 3-connected equimatchable factor-critical graphs with a given genus to a much larger class. For an extensive discussion of nowhere-dense and somewhere-dense graph classes see [26].

The third application of the main theorem is the description of the structure of k-connected factor-critical equimatchable graphs with respect to a minimum cut. Let $k \geq 3$ be a fixed integer and let G be a k-connected equimatchable factor-critical graph. We prove that if G has at least $2k+3$ vertices and a k-cut S such that such that $G-S$ has two components with sizes at least 3, then $G-S$ has exactly two components and both are complete graphs. Additionally, we prove that if $k \geq 4$, then all such graphs have independence number 2. Moreover, we provide also a characterisation of k-connected equimatchable factor-critical graphs with a k-cut S such that $G-S$ has a component with size at least k and a component with size 1 or 2. These results extend the previous result, which describes the structure of equimatchable factor-critical graphs with a cut-vertex or a 2-cut.

There are several lines of research which can be seen as a continuation of our work and in the following paragraphs we outline a few of them.

The characterisation of all planar 3-connected equimatchable graphs in [16] uses a special case of our main theorem. With our general version which does not require planarity, it may be possible to characterise for instance all toroidal 2-connected equimatchable factor-critical graphs. In particular, it can be readily observed that any such graph has at most 21 vertices. One may then proceed for example by considering cases according to the connectivity of the graph and using our characterisation of equimatchable factor-critical graphs with respect to minimum cuts. Another option would be to use our main theorem directly with the fact that any toroidal graph has a vertex of degree at most 6.

As can be seen from the results of Chapter 3, graphs with independence number 2 and equimatchable graphs are closely related. A particular open problem about graphs with independence number 2 is the following conjecture:

Conjecture 1 ([2, 38]). *Every graph G with independence number 2 has a $K_{\lceil |V(G)|/2 \rceil}$ as a minor.*

While Conjecture 1 would be implied by Hadwiger conjecture, so far it was confirmed only for graphs with connectivity at most $\lceil |V(G)/2| \rceil$. Note that the difficult cases in Conjecture 1 are difficult also in the characterisation of equimatchable factor-critical graphs. It remains to be seen whether extendability properties of graphs with independence number 2 can be useful either in investigation of the remaining open cases of Conjecture 1, or in providing a simpler proof of at least some of the known cases.

Regarding specific open problems, we propose the following three.
1) Are there only finitely many 3-connected bipartite equimatchable graphs embeddable in the surface of genus g with face-with at most 3?
2) Characterize all graphs with a perfect matching such that every matching of a graph misses at most 2 vertices.
3) If $\mathcal{C}$ is a nowhere-dense graph class, are there only finitely many 2-connected factor-critical equimatchable graphs in $\mathcal{C}$?

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